ON THE LAMN PROPERTY FOR CONTINUOUS OBSERVATIONS OF SOME DIFFUSION PROCESSES WITH JUMPS

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Abstract. In this paper, we consider a diffusion process with jumps whose drift and jump coefficient depend on an unknown parameter. We then give a self-contained proof of the local asymptotic mixed normality (LAMN) property when the process is observed continuously in a time interval $[0,T]$ as $T \to +\infty$, and derive, as a consequence, the local asymptotic normality (LAN) property in the ergodic case. For this, we give a proof of a Girsanov's theorem and a Central Limit theorem for a pure jump martingale. Our results could be viewed as a consequence of the LAMN property for semimartingales proved by Luschgy [15], using the Girsanov's theorem for semimartingales obtained in Jacod and Shiryaev [9], and the Central Limit theorem for semimartingales established by Sørensen [21] and Feigin [3]. The aim of this paper is to present a proof of these results without using this abstract semimartingale theory but integral equations with respect to Poisson random measures.

1. Introduction

On a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$ we consider a d-dimensional process X^θ solution to the following stochastic differential equation with jumps:

$$dX_t = a(\theta, X_t)dt + \sigma(X_t)dB_t + \int_{\mathbb{R}_0^d} c(\theta, X_{t-}, z)(p_\theta(dt, dz) - \nu_\theta(dz)dt), \quad t \geq 0, \tag{1.1}$$

where the initial condition X_0 is a random variable with finite second moment, $\mathbb{R}_0^d = \mathbb{R}^d \setminus \{0\}$, the unknown parameter θ belongs to an open subset Θ of $\mathbb{R}^k$, for some integer $k \geq 1$, $B = \{B_t\}_{t \geq 0}$ is a d-dimensional standard Brownian motion, and $p_\theta(dt, dz)$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_0^d$, independent of B with intensity measure $\nu_\theta(dz)dt = f(\theta, z)dzdt$. Here, $\nu_\theta(dz)$ is a Lévy measure on $\mathbb{R}_0^d$ such that $\int_{\mathbb{R}_0^d}(1 \wedge |z|^2)\nu_\theta(dz) < \infty$, for all $\theta \in \Theta$, and its Lévy density $f : \Theta \times \mathbb{R}^d \to \mathbb{R}_+$ is a Borel function such that $f(\theta, \{0\}) = 0$ and $\lambda(\theta) := \int_{\mathbb{R}^d} f(\theta, z)dz < \infty$, for all $\theta \in \Theta$, which implies that the number of jumps of X^θ is almost surely finite in finite time intervals. The measure $p_\theta(dt, dz) - \nu_\theta(dz)dt$ denotes the compensated Poisson random measure associated with $p_\theta(dt, dz)$.

The coefficients $a = (a_i)$ and $c = (c_i)$ are $\mathbb{R}^d$-valued Borel functions on $\Theta \times \mathbb{R}^d$ and $\Theta \times \mathbb{R}^d \times \mathbb{R}_0^d$, respectively, and $\sigma = (\sigma_{ij})$ is a $d \times d$ invertible Borel matrix on $\mathbb{R}^d$.

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We let $\{\mathcal{F}_t\}_{t\geq 0}$ denote the natural filtration generated by the Brownian motion and the Poisson random measure. By definition, the solution to equation (1.1) is a càdlàg and $\{\mathcal{F}_t\}$-adapted d-dimensional stochastic process $X^\theta = \{X_t\}_{t\geq 0}$ defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathrm{P})$ such that

$$X_t = X_0 + \int_0^t a(\theta, X_s)ds + \int_0^t \sigma(X_s)dB_s + \int_0^t \int_{\mathbb{R}_0^d} c(\theta, X_{s-}, z)(p_\theta(ds, dz) - \nu_\theta(dz)ds). \quad (1.2)$$

For any $\theta \in \Theta$, we denote by P_θ the probability measure induced by the solution X^θ of (1.1) on the canonical space $(D(\mathbb{R}^d), \mathcal{B}(\mathbb{R}^d))$, where $D(\mathbb{R}^d)$ denotes the space of càdlàg functions from $\mathbb{R}^d$ to $\mathbb{R}^d$, and $\mathcal{B}(\mathbb{R}^d)$ its associated Borel σ-algebra. Moreover, for any $T \geq 0$, we let P_θ^T denote the probability measure generated by the process $X^T = \{X_t : 0 \leq t \leq T\}$ solving equation (1.1) under the parameter θ on the measurable space $(D[0,T], \mathcal{B}[0,T])$. Therefore, P_θ^T is the restriction of P_θ to $\mathcal{F}_T$. For any $\theta \in \Theta$, we denote by E_θ the expectation with respect to the probability law P_θ, and $\xrightarrow{\mathrm{P}_\theta}$ and $\xrightarrow{\mathcal{L}(\mathrm{P}_\theta)}$ denote the convergence in P_θ-probability and the convergence in law under P_θ, respectively.

In this paper, we are interested in the statistical inference for $\theta \in \Theta$ on the basis of continuous-time observations of the process $X^T = \{X_t : 0 \leq t \leq T\}$ in the time interval $[0, T]$, as T tends to $+\infty$. Let us start by recalling the concepts on asymptotic statistical inference that we are interested in for our continuously observed parametric model.

We define the log-likelihood function of the family of probability measures $(\mathrm{P}_\theta^T)_{\theta\in\Theta}$ as

$$\ell_T(\theta) = \log \frac{d\mathrm{P}_\theta^T}{d\tilde{\mathrm{P}}^T},$$

where $\tilde{\mathrm{P}}^T$ is a probability measure on $(D[0,T], \mathcal{B}[0,T])$ satisfying that P_θ^T is absolutely continuous with respect to $\tilde{\mathrm{P}}^T$, for all $T \geq 0$ and $\theta \in \Theta$.

The score function, when it exists, is given by the gradient $\nabla_\theta \ell_T(\theta)$. We say that the score function is asymptotically normal if, for any $\theta \in \Theta$, there exists a $k \times k$ non-random diagonal matrix $\varphi_T(\theta)$ whose entries are strictly positive and tend to zero as $T \to \infty$, and a $k \times k$ positive definite random matrix $\Gamma(\theta)$, such that as $T \to \infty$,

$$\varphi_T(\theta)\nabla_\theta \ell_T(\theta) \xrightarrow{\mathcal{L}(\mathrm{P}_\theta)} \Gamma(\theta)^{1/2}\mathcal{N}\big(0, I_k\big), \quad (1.3)$$

where $\mathcal{N}(0, I_k)$ denotes a centered $\mathbb{R}^k$-valued Gaussian variable independent of $\Gamma(\theta)$ with identity covariance matrix I_k. In this case, the matrix $\Gamma(\theta)$ is called the asymptotic Fisher information matrix of the model.

The family of probability measures $(\mathrm{P}_\theta^T)_{\theta\in\Theta}$ is said to have the local asymptotic mixed normality (LAMN) property if for any $\theta \in \Theta$ and $u \in \mathbb{R}^k$, as $T \to \infty$,

$$\log \frac{d\mathrm{P}_{\theta+\varphi_T(\theta)u}^T}{d\mathrm{P}_\theta^T} \xrightarrow{\mathcal{L}(\mathrm{P}_\theta)} u^\mathsf{T}\Gamma(\theta)^{1/2}\mathcal{N}\big(0, I_k\big) - \frac{1}{2}u^\mathsf{T}\Gamma(\theta)u, \quad (1.4)$$

where $\mathcal{N}(0, I_k)$, $\varphi_T^{-1}(\theta)$, and $\Gamma(\theta)$ are as in (1.3). In this case, we say that the LAMN property holds with rate of convergence $\varphi_T^{-1}(\theta)$ and conditional covariance matrix $\Gamma(\theta)$. When the matrix $\Gamma(\theta)$ is deterministic, we say that the LAN property holds.

Observe that (1.4) is equivalent to

$$\log \frac{d\mathrm{P}^T_{\theta+\varphi_T(\theta)u}}{d\mathrm{P}^T_\theta} = \ell_T\left(\theta+\varphi_T(\theta)u\right) - \ell_T(\theta)$$
$$= u^\mathsf{T}\varphi_T(\theta)\nabla_\theta \ell_T(\theta) - \frac{1}{2}u^\mathsf{T}\Gamma(\theta)u + o_{\mathrm{P}_\theta}(1),$$

where $\varphi_T(\theta)\nabla_\theta\ell_T(\theta)$ converges in law to $\Gamma(\theta)^{1/2}\mathcal{N}(0, I_k)$ under P_θ as $T\to\infty$.

The aim of this paper is to revise sufficient conditions in order to have the asymptotic normality of the score function and the LAMN property for our diffusion model with jumps (1.1). This problem was addressed by Luschgy for semimartingales in [15], by using the Girsanov's theorem for semimartingales established by Jacod and Shiryaev (see [9, Theorem III.3.24]), and the Central Limit theorem for multivariate martingales proved by Sørensen (see [21, Theorem A.1]), as an extension of the Central Limit theorem for one-dimensional martingales [3, Theorem 2] by Feigin. We remark that the stochastic process with jumps (1.2) is a semimartingale. Therefore, Luschgy's theorem applies and one can derive sufficient conditions on the coefficients in order to have the LAMN property. The aim of this paper is to present a self-contained proof of the LAMN property for the solution X^θ of (1.1) by following the proof of Luschgy but proving a Girsanov's theorem and a Central Limit theorem without using the fact that we have a semimartingale, but using the integral equation (1.2). We then deduce the LAN property with an explicit asymptotic Fisher information matrix in the case where the process X^θ is ergodic. To obtain our results, the first step consists in transforming equation (1.1) into a new stochastic differential equation with jumps driven by a compound Poisson process. Notice that this approach was also employed by Sørensen in [21]. One of the motivations of writing this paper is that we are investigating in a further paper the LAMN property for the stochastic differential equations with jumps (1.1) with discrete observations in a time interval $[0,T]$ as $T\to\infty$, which has never been addressed in the literature. For this, we think it is essential to first understand the proof of this property in the continuously observed case but without using the asbtract semimartingale theory, but integral equations with respect to Poisson random measures.

As is well known, the local asymptotic normality (LAN) property is a fundamental concept in asymptotic theory of statistics, which was introduced by Le Cam (see [14]) and extended by Jeganathan to the local asymptotic mixed normality (LAMN) property (see [10]). Such properties have largely been studied in the literature. More precisely, when the process is observed continuously, the LAN property was studied by Kutoyants in [13] for ergodic diffusion processes. As mentioned before, Luschgy in [15] proved the LAMN property for semimartingales. In the case of discrete observations at high frequency, Gobet in [5] and [6] obtained the LAMN and LAN properties for multidimensional diffusion processes. For this, techniques of the Malliavin calculus were applied in order to derive a representation of the log-likelihood function in terms of a conditional expectation. Moreover, in the same direction, Gobet and Gloter in [4] showed that the LAMN property is satisfied for hidden processes. The LAN property is also addressed for stable processes by Woerner in [24] and for normal inverse Gaussian Lévy processes by Kawai and Masuda in [11]. Recently, using tools of the Malliavin calculus, Clément and *al.* in [2] have established the LAMN property for a parametric model with jumps in which the unknown parameter determines the amplitude of the jumps. Last but not least, Sørensen establishes the asymptotic normality of the score function for our diffusion process (1.1) as a consequence of the Girsanov's theorem for semimartingales established

by Jacod and Mémin in [8], and studies asymptotic properties of the maximum likelihood estimator in some particular cases.

This paper is organized as follows. In Section 2, we provide sufficient conditions and prove the asymptotic normality of the score function as well as the LAMN property for the stochastic differential equation with jumps (1.1). Studying the LAMN property from continuous observations is based on the Girsanov's theorem for equivalent probability measures. Therefore, we will give a proof of this fundamental result in Section 3. A Central Limit theorem for a stochastic integral with respect to a compensated Poisson random measure is also needed and this will be proved in Section 4. The proof of the LAN property in the ergodic case as a consequence of the LAMN property is given in Section 5. Finally, Section 6 deals with a concrete example where the LAMN and LAN properties are satisfied and the maximum likelihood estimator is asymptotically efficient in some particular cases.

2. LAMN property for jump diffusions

The aim of this section is to give sufficient conditions in order to have the asymptotic normality of the score function and the LAMN property for our stochastic differential equation with jumps (1.1). To this purpose, let us first recall the result on the existence and uniqueness of the solution to our integral equation (1.2), that is,

$$X_t = X_0 + \int_0^t a(\theta, X_s)ds + \int_0^t \sigma(X_s)dB_s + \int_0^t \int_{\mathbb{R}_0^d} c(\theta, X_{s-}, z)(p_\theta(ds, dz) - \nu_\theta(dz)ds).$$

Consider the following Lipschitz continuity and linear growth conditions on the coefficients.

(A1) There exist a constant $L > 0$ and a function $\zeta : \mathbb{R}_0^d \to \mathbb{R}_+$ satisfying that $\int_{\mathbb{R}_0^d} \zeta^2(z)\nu_\theta(dz) < \infty$, for all $\theta \in \Theta$, such that for any $x, y \in \mathbb{R}^d$, $z \in \mathbb{R}_0^d$, and $\theta \in \Theta$,

$$|a(\theta, x) - a(\theta, y)| + |\sigma(x) - \sigma(y)| \leq L|x - y|, \quad |a(\theta, x)| \leq L(1 + |x|),$$
$$|c(\theta, x, z) - c(\theta, y, z)| \leq \zeta(z)|x - y|, \quad |c(\theta, x, z)| \leq \zeta(z)(1 + |x|).$$

Theorem 2.1. [9, Theorem III-2-32] *Under condition* **(A1)***, there exists a unique càdlàg and adapted process $X^\theta = \{X_t\}_{t\geq 0}$ solution to equation* (1.1) *on the filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathrm{P})$*. Moreover, for any fixed $p > 0$ and $T > 0$, there exists a constant $C_{p,T} > 0$ such that for all $t_0 \in (0, T]$ and $t \in [t_0, T]$,*

$$\mathrm{E}\left[\sup_{t_0\leq s\leq t} |X_s - X_{t_0}|^p\right] \leq C_{p,T}(t - t_0)^{(p/2)\wedge 1}\mathrm{E}\left[(1 + |X_{t_0}|^2)^{p/2}\right]. \tag{2.1}$$

Let us now proceed as in [21] to transform our equation (1.1) into a new stochastic differential equation with jumps driven by a compound Poisson process via a change of variables.

For each $(\theta, x) \in \Theta \times \mathbb{R}^d$ fixed, we assume that the mapping $z \in \mathbb{R}_0^d \mapsto y = c(\theta, x, z) \in \mathrm{Im}(c) \cap \mathbb{R}_0^d$ has a continuous differentiable inverse $y \in \mathrm{Im}(c) \cap \mathbb{R}_0^d \mapsto z = c^{-1}(\theta, x, y) \in \mathbb{R}_0^d$ with Jacobian matrix $J(\theta, x, y)$ such that $\det(J(\theta, x, y)) \neq 0$, for all $y \in \mathrm{Im}(c) \cap \mathbb{R}_0^d$.

Set $\Psi(\theta, x, y) = f(\theta, c^{-1}(\theta, x, y))|\det(J(\theta, x, y))|$, and suppose that for all $(\theta, x) \in \Theta \times \mathbb{R}^d$,

$$\int_{\mathbb{R}_0^d} c(\theta, x, z)\nu_\theta(dz) < +\infty.$$

Then equation (1.1) can be rewritten as follows (see [9, Proposition II.1.28])

$$dX_t = b(\theta, X_t)dt + \sigma(X_t)dB_t + \int_{\mathrm{Im}(c)\cap\mathbb{R}_0^d} yN_\theta(dt, dy), \tag{2.2}$$

where the function $b : \Theta \times \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$b(\theta, X_t) = a(\theta, X_t) - \int_{\mathbb{R}_0^d} c(\theta, X_{t-}, z)\nu_\theta(dz) = a(\theta, X_t) - \int_{\mathbb{R}_0^d} y\mu_\theta(dy),$$

and $N_\theta(dt, dy)$ is a new Poisson random measure on $\mathbb{R}_+ \times \mathrm{Im}(c) \cap \mathbb{R}_0^d$ with intensity measure $\mu_\theta(dy)dt = \Psi(\theta, X_{t-}, y)dydt$. Observe that $\lambda(\theta) = \int_{\mathbb{R}_0^d} \mu_\theta(dy) < \infty$, P_θ-a.s. Thus, $N_\theta(dt, dy) - \mu_\theta(dy)dt$ is a new compensated Poisson random measure associated with $N_\theta(dt, dy)$.

To simplify the exposition we assume that $\mathrm{Im}(c) = \mathbb{R}^d$.

In order to have the asymptotic normality of the score function and the LAMN property, we assume that there exists a $k \times k$ non-random diagonal matrix $\varphi_T(\theta)$ whose entries are strictly positive and tend to zero as $T \to \infty$, and such that the following conditions hold.

(A2) The functions $a(\theta, x)$ and $\Psi(\theta, x, y)$ are differentiable with respect to θ, and the functions $\Psi(\theta, x, y)$ and $\nabla_\theta \Psi(\theta, x, y)$ are continuous in θ.

(A3) There exists a $k \times k$ random symmetric non-negative definite random matrix $\Gamma_1(\theta)$ such that for all $u \in \mathbb{R}^k$, as $T \to \infty$,

$$\int_0^T |\sigma^{-1}(X_t)\nabla_\theta b(\theta, X_t)\varphi_T(\theta)u|^2 dt \xrightarrow{\mathrm{P}_\theta} u^\mathsf{T}\Gamma_1(\theta)u,$$

uniformly in $\theta \in \Theta$.

(A4) There exists a $k \times k$ random symmetric non-negative definite random matrix $\Gamma_2(\theta)$ such that for all $u \in \mathbb{R}^k$, as $T \to \infty$,

$$\int_0^T \int_{\mathbb{R}_0^d} \left(u^\mathsf{T}\varphi_T(\theta)\nabla_\theta \ln\left(\Psi(\theta, X_{t-}, y)\right)\right)^2 \mu_\theta(dy)dt \xrightarrow{\mathrm{P}_\theta} u^\mathsf{T}\Gamma_2(\theta)u,$$

uniformly in $\theta \in \Theta$.

(A5) For every $T \geq 0$ and $\theta \in \Theta$,

$$\begin{aligned}
&\mathrm{E}_\theta\left[\int_0^T (\nabla_\theta b(\theta, X_t))^\mathsf{T}(\sigma^{-1}(X_t))^\mathsf{T}\sigma^{-1}(X_t)\nabla_\theta b(\theta, X_t)dt\right] \\
&+ \mathrm{E}_\theta\left[\int_0^T \int_{\mathbb{R}_0^d} (\nabla_\theta \ln\left(\Psi(\theta, X_{t-}, y)\right))^\mathsf{T}\nabla_\theta \ln\left(\Psi(\theta, X_{t-}, y)\right)\, \mu_\theta(dy)dt\right] < \infty,
\end{aligned} \tag{2.3}$$

and the matrix $\Gamma(\theta) := \Gamma_1(\theta) + \Gamma_2(\theta)$ is positive definite.

(A6) For all $\epsilon > 0$ and $u \in \mathbb{R}^k$,

$$\int_0^T \int_{\mathbb{R}_0^d} \mathrm{E}_\theta\left[\left(u^\mathsf{T}\varphi_T(\theta)\nabla_\theta \ln\left(\Psi(\theta, X_{t-}, y)\right)\right)^2 \mathbf{1}_{\left\{\left|u^\mathsf{T}\varphi_T(\theta)\nabla_\theta \ln(\Psi(\theta, X_{t-}, y))\right| > \epsilon\right\}} \Psi(\theta, X_{t-}, y)\right] dydt$$

tends to 0 as T tends to ∞, uniformly in $\theta \in \Theta$.

(A7) For all $u \in \mathbb{R}^k$, as $T \to \infty$,

$$\int_0^T \left|\sigma^{-1}(X_t)\left(b(\theta + \varphi_T(\theta)u, X_t) - b(\theta, X_t) - \nabla_\theta b(\theta, X_t)\varphi_T(\theta)u\right)\right|^2 dt \xrightarrow{\mathrm{P}_\theta} 0,$$

uniformly in $\theta \in \Theta$.

(A8) For all $u \in \mathbb{R}^k$, as $T \to \infty$,

$$\int_0^T \int_{\mathbb{R}_0^d} \left(\frac{\Psi(\theta + \varphi_T(\theta)u, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 - u^\mathsf{T}\varphi_T(\theta)\nabla_\theta \ln\left(\Psi(\theta, X_{t-}, y)\right)\right)^2 \mu_\theta(dy)dt \xrightarrow{\mathrm{P}_\theta} 0,$$

uniformly in $\theta \in \Theta$.

We first state the asymptotic normality of the score function.

Theorem 2.2. *Suppose that conditions* **(A1)**-**(A6)** *are fulfilled. Then the score function is asymptotically normal uniformly for all $\theta \in \Theta$ with asymptotic Fisher information matrix $\Gamma(\theta)$. That is, as $T \to \infty$,*

$$\varphi_T(\theta)\nabla_\theta \ell_T(\theta) \xrightarrow{\mathcal{L}(\mathrm{P}_\theta)} \Gamma(\theta)^{1/2}\mathcal{N}\big(0, I_k\big),$$

uniformly in $\theta \in \Theta$, where $\mathcal{N}(0, I_k)$ is a centered $\mathbb{R}^k$-valued Gaussian variable independent of $\Gamma(\theta)$.

Proof. By Girsanov's Theorem 3.1, the log-likelihood function is given by

$$\begin{aligned}
\ell_T(\theta) &= \log \frac{d\mathrm{P}_\theta^T}{d\mathrm{P}_{\theta_0}^T} \\
&= \int_0^T \sigma^{-1}(X_t)\left(b(\theta, X_t) - b(\theta_0, X_t)\right) \cdot dB_t - \frac{1}{2}\int_0^T \left|\sigma^{-1}(X_t)\left(b(\theta, X_t) - b(\theta_0, X_t)\right)\right|^2 dt \\
&\quad + \int_0^T \int_{\mathbb{R}_0^d} \ln \frac{\Psi(\theta, X_{t-}, y)}{\Psi(\theta_0, X_{t-}, y)} N_{\theta_0}(dt, dy) - \int_0^T \int_{\mathbb{R}_0^d} \Big(\frac{\Psi(\theta, X_{t-}, y)}{\Psi(\theta_0, X_{t-}, y)} - 1\Big)\mu_{\theta_0}(dy)dt,
\end{aligned}$$

for any $\theta_0 \in \Theta$.

Therefore, the score function is given by

$$\begin{aligned}
\nabla_\theta \ell_T(\theta) &= \int_0^T \sigma^{-1}(X_t)\nabla_\theta b(\theta, X_t) \cdot \big(dB_t - \sigma^{-1}(X_t)\left(b(\theta, X_t) - b(\theta_0, X_t)\right)\big) \\
&\quad + \int_0^T \int_{\mathbb{R}_0^d} \nabla_\theta \ln\left(\Psi(\theta, X_{t-}, y)\right)\left(N_\theta(dt, dy) - \mu_\theta(dy)dt\right).
\end{aligned}$$

Now, by the classical Girsanov's Theorem, the process $W = (W_t, 0 \le t \le T)$ defined as

$$W_t = B_t - \int_0^t \sigma^{-1}(X_s)\left(b(\theta, X_s) - b(\theta_0, X_s)\right) ds$$

is an $(\mathcal{F}_t, 0 \le t \le T)$-Brownian motion under P_θ. Therefore, under P_θ,

$$\nabla_\theta \ell_T(\theta) = \int_0^T \sigma^{-1}(X_t)\nabla_\theta b(\theta, X_t) \cdot dW_t$$

$$+\int_0^T \int_{\mathbb{R}_0^d} \nabla_\theta \ln\left(\Psi(\theta, X_{t-}, y)\right)\left(N_\theta(dt, dy) - \mu_\theta(dy)dt\right).$$

Finally, observe that by (2.3), $\nabla_\theta \ell_T(\theta)$ is a square integrable P_θ-martingale. Then, from the Central Limit Theorems [19, Theorem 1.50] and 4.1, we obtain the desired result. □

We next state the LAMN property for the jump diffusion process solution to (1.1) on the time interval $[0, T]$.

Theorem 2.3. *Suppose that conditions* **(A1)**-**(A8)** *are fulfilled. Then the LAMN property holds uniformly for all $\theta \in \Theta$ with rate of convergence $\varphi_T^{-1}(\theta)$ and asymptotic Fisher information matrix $\Gamma(\theta)$. That is, for all $u \in \mathbb{R}^k$, as $T \to \infty$,*

$$\log \frac{d\mathrm{P}^T_{\theta+\varphi_T(\theta)u}}{d\mathrm{P}^T_\theta} \overset{\mathcal{L}(\mathrm{P}_\theta)}{\longrightarrow} u^\mathsf{T}\Gamma(\theta)^{1/2}\mathcal{N}(0, I_k) - \frac{1}{2}u^\mathsf{T}\Gamma(\theta)u,$$

uniformly in $\theta \in \Theta$, where $\mathcal{N}(0, I_k)$ is a centered $\mathbb{R}^k$-valued Gaussian variable independent of $\Gamma(\theta)$.

Remark 2.4. We observe that conditions **(A3)** and **(A4)**, **(A7)** and **(A8)** are the same as conditions **(L)**, **(D.1)** and **(D.3)** of Luschgy [15], respectively. Moreover, condition **(A6)** implies his condition **(J.3)**. Therefore, Theorem 2.3 is a consequence of Theorem 1 of Luschgy [15]. Observe that we do not need his condition **(J.1)** because we are assuming the stronger condition **(A6)**. Will use this condition in order to obtain a Central Limit theorem for a stochastic integral with respect to a compensated Poisson random measure (see Theorem 4.1 below). Finally, remark that we do not need his convergence condition **(R)** since he is using this condition in order to establish the stable convergence which is stronger than the convergence in distribution.

Proof. Fix $u \in \mathbb{R}^k$ and $\theta \in \Theta$, and apply Girsanov's Theorem 3.1 with $\theta_0 = \theta + \varphi_T(\theta)u$, to get that

$$\log \frac{d\mathrm{P}^T_{\theta_0}}{d\mathrm{P}^T_\theta} = L_T^c + L_T^d, \tag{2.4}$$

where the continuous part L_T^c and the discontinuous part L_T^d are respectively given by

$$L_T^c = \int_0^T \sigma^{-1}(X_t)\left(b(\theta_0, X_t) - b(\theta, X_t)\right) \cdot dB_t - \frac{1}{2}\int_0^T \left|\sigma^{-1}(X_t)\left(b(\theta_0, X_t) - b(\theta, X_t)\right)\right|^2 dt,$$

$$L_T^d = \int_0^T \int_{\mathbb{R}_0^d} \ln \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} N_\theta(dt, dy) - \int_0^T \int_{\mathbb{R}_0^d} \Big(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1\Big)\mu_\theta(dy)dt.$$

First, we treat the continuous part L_T^c. Adding and substracting the vector $\nabla_\theta b(\theta, X_t)\varphi_T(\theta)u$, we get that

$$\begin{aligned}
L_T^c &= \int_0^T \sigma^{-1}(X_t)\left(b(\theta_0, X_t) - b(\theta, X_t) - \nabla_\theta b(\theta, X_t)\varphi_T(\theta)u\right) \cdot dB_t \\
&+ \int_0^T \sigma^{-1}(X_t)\nabla_\theta b(\theta, X_t)\varphi_T(\theta)u \cdot dB_t \\
&- \frac{1}{2}\int_0^T \left|\sigma^{-1}(X_t)\left(b(\theta_0, X_t) - b(\theta, X_t) - \nabla_\theta b(\theta, X_t)\varphi_T(\theta)u\right)\right|^2 dt
\end{aligned}$$

$$
-\frac{1}{2}\int_0^T |\sigma^{-1}(X_t)\nabla_\theta b(\theta,X_t)\varphi_T(\theta)u|^2 dt
$$

$$
-\int_0^T u^\top\varphi_T(\theta)\left(\nabla_\theta b(\theta,X_t)\right)^\top(\sigma^{-1}(X_t))^\top\sigma^{-1}(X_t)\left(b(\theta_0,X_t)-b(\theta,X_t)-\nabla_\theta b(\theta,X_t)\varphi_T(\theta)u\right)dt.
$$

From hypothesis **(A3)** and the multidimensional Central Limit theorem for stochastic integrals with respect to Brownian motion (see [19, Theorem 1.50]), we get that as $T\to\infty$,

$$
\int_0^T \sigma^{-1}(X_t)\nabla_\theta b(\theta,X_t)\varphi_T(\theta)u\cdot dB_t \overset{\mathcal{L}(\mathrm{P}_\theta)}{\longrightarrow} u^\top\Gamma_1(\theta)^{1/2}\mathcal{N}(0,I_k)
$$

uniformly in $\theta\in\Theta$, where $\mathcal{N}(0,I_k)$ is a centered $\mathbb{R}^k$-valued Gaussian variable independent of $\Gamma_1(\theta)$.

Now, by hypothesis **(A7)**, the quadratic variation of the martingale

$$
\int_0^T \sigma^{-1}(X_t)\left(b(\theta_0,X_t)-b(\theta,X_t)-\nabla_\theta b(\theta,X_t)\varphi_T(\theta)u\right)\cdot dB_t
$$

tends to zero in P_θ-probability as $T\to\infty$ uniformly in $\theta\in\Theta$. Thus, so does the martingale.

Now, using the Cauchy-Schwarz inequality, one gets that

$$
\begin{aligned}
&\left|\int_0^T u^\top\varphi_T(\theta)\left(\nabla_\theta b(\theta,X_t)\right)^\top(\sigma^{-1}(X_t))^\top\sigma^{-1}(X_t)\left(b(\theta_0,X_t)-b(\theta,X_t)-\partial_\theta b(\theta,X_t)\varphi_T(\theta)u\right)dt\right|\\
&\quad\le\left(\int_0^T |\sigma^{-1}(X_t)\nabla_\theta b(\theta,X_t)\varphi_T(\theta)u|^2 dt\right)^{1/2}\\
&\quad\quad\times\left(\int_0^T \left|\sigma^{-1}(X_t)\left(b(\theta_0,X_t)-b(\theta,X_t)-\nabla_\theta b(\theta,X_t)\varphi_T(\theta)u\right)\right|^2 dt\right)^{1/2}.
\end{aligned}
$$

Therefore, from **(A3)** and **(A7)**, we conclude that as $T\to\infty$,

$$
L_T^c \overset{\mathcal{L}(\mathrm{P}_\theta)}{\longrightarrow} u^\top\Gamma_1(\theta)^{1/2}\mathcal{N}(0,I_k)-\frac{1}{2}u^\top\Gamma_1(\theta)u, \tag{2.5}
$$

uniformly in $\theta\in\Theta$.

Next, we study the term L_T^d. Adding and substracting the terms $u^\top\varphi_T(\theta)\nabla_\theta\ln\left(\Psi(\theta,X_{t-},y)\right)$ and $\frac{1}{2}\left(u^\top\varphi_T(\theta)\nabla_\theta\ln\left(\Psi(\theta,X_{t-},y)\right)\right)^2$, we get that

$$
\begin{aligned}
L_T^d&=\int_0^T\int_{\mathbb{R}_0^d} u^\top\varphi_T(\theta)\nabla_\theta\ln\left(\Psi(\theta,X_{t-},y)\right)\left(N_\theta(dt,dy)-\mu_\theta(dy)dt\right)\\
&-\frac{1}{2}\int_0^T\int_{\mathbb{R}_0^d}\left(u^\top\varphi_T(\theta)\nabla_\theta\ln\left(\Psi(\theta,X_{t-},y)\right)\right)^2\mu_\theta(dy)dt\\
&+\int_0^T\int_{\mathbb{R}_0^d}\left(\frac{\Psi(\theta_0,X_{t-},y)}{\Psi(\theta,X_{t-},y)}-1-u^\top\varphi_T(\theta)\nabla_\theta\ln\left(\Psi(\theta,X_{t-},y)\right)\right)\left(N_\theta(dt,dy)-\mu_\theta(dy)dt\right)\\
&+\int_0^T\int_{\mathbb{R}_0^d}\left(\ln\frac{\Psi(\theta_0,X_{t-},y)}{\Psi(\theta,X_{t-},y)}-\frac{\Psi(\theta_0,X_{t-},y)}{\Psi(\theta,X_{t-},y)}+1+\frac{1}{2}\left(u^\top\varphi_T(\theta)\nabla_\theta\ln\left(\Psi(\theta,X_{t-},y)\right)\right)^2\right)N_\theta(dt,dy)\\
&-\frac{1}{2}\int_0^T\int_{\mathbb{R}_0^d}\left(u^\top\varphi_T(\theta)\nabla_\theta\ln\left(\Psi(\theta,X_{t-},y)\right)\right)^2\left(N_\theta(dt,dy)-\mu_\theta(dy)dt\right).
\end{aligned}
$$

By hypothesis **(A8)**, the quadratic characteristic of the martingale

$$\int_0^T \int_{\mathbb{R}_0^d} \left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 - u^\intercal \varphi_T(\theta) \nabla_\theta \ln \left(\Psi(\theta, X_{t-}, y)\right) \right) (N_\theta(dt, dy) - \mu_\theta(dy)dt)$$

tends to zero in P_θ-probability as $T \to \infty$ uniformly in $\theta \in \Theta$. Thus, so does the martingale.

Then, appealing to Lemmas 2.5 and 2.6 below and the Central Limit Theorem 4.1, together with hypotheses **(A4)**, **(A6)** and **(A8)**, we conclude that as $T \to \infty$,

$$L_T^d \xrightarrow{\mathcal{L}(\mathrm{P}_\theta)} u^\intercal \Gamma_2(\theta)^{1/2} \mathcal{N}(0, I_k) - \frac{1}{2} u^\intercal \Gamma_2(\theta) u, \tag{2.6}$$

uniformly in $\theta \in \Theta$, where $\mathcal{N}(0, I_k)$ is a centered $\mathbb{R}^k$-valued Gaussian variable independent of $\Gamma_2(\theta)$.

Finally, substituting (2.5) and (2.6) into (2.4), we conclude the desired proof. □

We are left to prove the following Lemmas.

Lemma 2.5. *Assume that the function $\psi(\theta, x, y)$ is differentiable with respect to θ and hypotheses* **(A4)** *and* **(A6)** *hold. Then, as $T \to \infty$,*

$$\int_0^T \int_{\mathbb{R}_0^d} \left(u^\intercal \varphi_T(\theta) \nabla_\theta \ln \left(\Psi(\theta, X_{t-}, y)\right) \right)^2 (N_\theta(dt, dy) - \mu_\theta(dy)dt) \xrightarrow{\mathrm{P}_\theta} 0,$$

uniformly in $\theta \in \Theta$.

Proof. First observe that hypothesis **(A6)** implies that for all $\epsilon > 0$ and $u \in \mathbb{R}^k$, as $T \to \infty$, the martingale

$$\int_0^T \int_{\mathbb{R}_0^d} \left(u^\intercal \varphi_T(\theta) \nabla_\theta \ln \left(\Psi(\theta, X_{t-}, y)\right) \right)^2 \mathbf{1}_{\left\{\left|u^\intercal \varphi_T(\theta) \nabla_\theta \ln(\Psi(\theta, X_{t-}, y))\right| > \epsilon\right\}} (N_\theta(dt, dy) - \mu_\theta(dy)dt)$$

tends to zero in P_θ-probability uniformly in $\theta \in \Theta$.

Furthermore, for all $\epsilon > 0$ and $u \in \mathbb{R}^k$, the quadratic characteristic of the martingale

$$\int_0^T \int_{\mathbb{R}_0^d} \left(u^\intercal \varphi_T(\theta) \nabla_\theta \ln \left(\Psi(\theta, X_{t-}, y)\right) \right)^2 \mathbf{1}_{\left\{\left|u^\intercal \varphi_T(\theta) \nabla_\theta \ln(\Psi(\theta, X_{t-}, y))\right| \leq \epsilon\right\}} (N_\theta(dt, dy) - \mu_\theta(dy)dt)$$

equals

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}_0^d} \left(u^\intercal \varphi_T(\theta) \nabla_\theta \ln \left(\Psi(\theta, X_{t-}, y)\right) \right)^4 \mathbf{1}_{\left\{\left|u^\intercal \varphi_T(\theta) \nabla_\theta \ln(\Psi(\theta, X_{t-}, y))\right| \leq \epsilon\right\}} \mu_\theta(dy)dt \\ &\leq \epsilon^2 \int_0^T \int_{\mathbb{R}_0^d} \left(u^\intercal \varphi_T(\theta) \nabla_\theta \ln \left(\Psi(\theta, X_{t-}, y)\right) \right)^2 \mathbf{1}_{\left\{\left|u^\intercal \varphi_T(\theta) \nabla_\theta \ln(\Psi(\theta, X_{t-}, y))\right| \leq \epsilon\right\}} \mu_\theta(dy)dt. \end{aligned}$$

Therefore, assumption **(A4)** concludes the proof of the lemma. □

Lemma 2.6. *Assume that the function $\psi(\theta, x, y)$ is differentiable with respect to θ and that hypotheses* **(A4)**, **(A6)**, *and* **(A8)** *hold. Then, as $T \to \infty$,*

$$\int_0^T \int_{\mathbb{R}_0^d} \left(\ln \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} + 1 + \frac{1}{2} \left(u^\intercal \varphi_T(\theta) \nabla_\theta \ln \left(\Psi(\theta, X_{t-}, y)\right) \right)^2 \right) N_\theta(dt, dy)$$

tends to zero in P_θ-probability uniformly in $\theta \in \Theta$.

Proof. Consider the function $f(y-1) = \ln(y) - (y-1) + \frac{1}{2}(y-1)^2$ defined for all $y > 0$. Then, for all x,

$$\ln(y) - y + 1 + \frac{1}{2}x^2 = f(y-1) - \frac{1}{2}((y-1)^2 - x^2).$$

Therefore,

$$\int_0^T \int_{\mathbb{R}_0^d} \left(\ln \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} + 1 + \frac{1}{2} \left(u^\mathsf{T} \varphi_T(\theta) \nabla_\theta \ln \left(\Psi(\theta, X_{t-}, y) \right) \right)^2 \right) N_\theta(dt, dy)$$
$$= \int_0^T \int_{\mathbb{R}_0^d} f \left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right) N_\theta(dt, dy)$$
$$- \frac{1}{2} \int_0^T \int_{\mathbb{R}_0^d} \left\{ \left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right)^2 - \left(u^\mathsf{T} \varphi_T(\theta) \nabla_\theta \ln \left(\Psi(\theta, X_{t-}, y) \right) \right)^2 \right\} N_\theta(dt, dy).$$

Now, observe that for any $T \geq 0$,

$$\mathrm{E}_\theta \left[\left| \int_0^T \int_{\mathbb{R}_0^d} \left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 - u^\mathsf{T} \varphi_T(\theta) \nabla_\theta \ln \left(\Psi(\theta, X_{t-}, y) \right) \right)^2 N_\theta(dt, dy) \right| \right]$$
$$= \mathrm{E}_\theta \left[\sum_{0 \leq s \leq T : \Delta X_s \neq 0} \left(\frac{\Psi(\theta_0, X_{s-}, \Delta X_s)}{\Psi(\theta, X_{s-}, \Delta X_s)} - 1 - u^\mathsf{T} \varphi_T(\theta) \nabla_\theta \ln \left(\Psi(\theta, X_{s-}, \Delta X_s) \right) \right)^2 \right]$$
$$= \mathrm{E}_\theta \left[\int_0^T \int_{\mathbb{R}_0^d} \left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 - u^\mathsf{T} \varphi_T(\theta) \nabla_\theta \ln \left(\Psi(\theta, X_{t-}, y) \right) \right)^2 \mu_\theta(dy) dt \right],$$

where, $\Delta X_s = X_s - X_{s-}$ denotes the jump of X_s at time s.

Therefore, the process $\int_0^T \int_{\mathbb{R}_0^d} \left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 - u^\mathsf{T} \varphi_T(\theta) \nabla_\theta \ln(\Psi(\theta, X_{t-}, y)) \right)^2 N_\theta(dt, dy)$ is dominated in the sense of Lenglart by its compensator process

$$\int_0^T \int_{\mathbb{R}_0^d} \left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 - u^\mathsf{T} \varphi_T(\theta) \nabla_\theta \ln \left(\Psi(\theta, X_{t-}, y) \right) \right)^2 \mu_\theta(dy) dt,$$

for any $T \geq 0$. Thus, by Lenglart's inequality [9, Lemma 3.30], we have that for all $T \geq 0$ and $\epsilon, \eta > 0$,

$$\mathrm{P}_\theta \left(\left| \int_0^T \int_{\mathbb{R}_0^d} \left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 - u^\mathsf{T} \varphi_T(\theta) \nabla_\theta \ln \left(\Psi(\theta, X_{t-}, y) \right) \right)^2 N_\theta(dt, dy) \right| \geq \epsilon \right)$$
$$\leq \frac{\eta}{\epsilon} + \mathrm{P}_\theta \left(\int_0^T \int_{\mathbb{R}_0^d} \left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 - u^\mathsf{T} \varphi_T(\theta) \nabla_\theta \ln \left(\Psi(\theta, X_{t-}, y) \right) \right)^2 \mu_\theta(dy) dt \geq \eta \right).$$

Hence, hypothesis **(A8)** implies that for all $u \in \mathbb{R}^k$, as $T \to \infty$,

$$\int_0^T \int_{\mathbb{R}_0^d} \left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 - u^\mathsf{T} \varphi_T(\theta) \nabla_\theta \ln \left(\Psi(\theta, X_{t-}, y) \right) \right)^2 N_\theta(dt, dy)$$

tends to zero in P_θ-probability uniformly in $\theta \in \Theta$.

Thus, from hypothesis **(A4)** and the equality $a^2 - b^2 = (a-b)^2 + 2b(a-b)$, we conclude that for all $u \in \mathbb{R}^k$, as $T \to \infty$,

$$\int_0^T \int_{\mathbb{R}_0^d} \left\{ \left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right)^2 - \left(u^\top \varphi_T(\theta) \nabla_\theta \ln\left(\Psi(\theta, X_{t-}, y)\right) \right)^2 \right\} N_\theta(dt, dy) \tag{2.7}$$

tends to zero in P_θ-probability uniformly in $\theta \in \Theta$.

We next show that for every $\epsilon > 0$, as $T \to \infty$,

$$\int_0^T \int_{\mathbb{R}_0^d} f\left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right) \mathbf{1}_{\left\{ \left| \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right| > \epsilon \right\}} N_\theta(dt, dy) \xrightarrow{\mathrm{P}_\theta} 0, \tag{2.8}$$

uniformly in $\theta \in \Theta$.

Observe that

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}_0^d} f\left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right) \mathbf{1}_{\left\{ \left| \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right| > \epsilon \right\}} N_\theta(dt, dy) \\ &= \sum_{0 \le s \le T : \Delta X_s \neq 0} f\left(\frac{\Psi(\theta_0, X_{s-}, \Delta X_s)}{\Psi(\theta, X_{s-}, \Delta X_s)} - 1 \right) \mathbf{1}_{\left\{ \left| \frac{\Psi(\theta_0, X_{s-}, \Delta X_s)}{\Psi(\theta, X_{s-}, \Delta X_s)} - 1 \right| > \epsilon \right\}}, \end{aligned}$$

and

$$\int_0^T \int_{\mathbb{R}_0^d} \mathbf{1}_{\left\{ \left| \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right| > \epsilon \right\}} N_\theta(dt, dy) = \sum_{0 \le s \le T : \Delta X_s \neq 0} \mathbf{1}_{\left\{ \left| \frac{\Psi(\theta_0, X_{s-}, \Delta X_s)}{\Psi(\theta, X_{s-}, \Delta X_s)} - 1 \right| > \epsilon \right\}}.$$

Hence, for all $a > 0$,

$$\begin{aligned} &\left\{ \left| \int_0^T \int_{\mathbb{R}_0^d} f\left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right) \mathbf{1}_{\left\{ \left| \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right| > \epsilon \right\}} N_\theta(dt, dy) \right| > a \right\} \\ &\subset \left\{ \int_0^T \int_{\mathbb{R}_0^d} \mathbf{1}_{\left\{ \left| \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right| > \epsilon \right\}} N_\theta(dt, dy) \geq 1 \right\}. \end{aligned}$$

Therefore, in order to prove (2.8), it suffices to show that for every $\epsilon > 0$, as $T \to \infty$,

$$\int_0^T \int_{\mathbb{R}_0^d} \mathbf{1}_{\left\{ \left| \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right| > \epsilon \right\}} N_\theta(dt, dy) \xrightarrow{\mathrm{P}_\theta} 0, \tag{2.9}$$

uniformly in $\theta \in \Theta$.

For this, we write

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}_0^d} \mathbf{1}_{\left\{ \left| \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right| > \epsilon \right\}} N_\theta(dt, dy) \\
&\leq \int_0^T \int_{\mathbb{R}_0^d} \mathbf{1}_{\left\{ \left| u^\intercal \varphi_T(\theta) \nabla_\theta \ln(\Psi(\theta, X_{t-}, y)) \right| > \frac{\epsilon}{2} \right\}} N_\theta(dt, dy) \\
&\qquad + \int_0^T \int_{\mathbb{R}_0^d} \mathbf{1}_{\left\{ \left| \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 - u^\intercal \varphi_T(\theta) \nabla_\theta \ln(\Psi(\theta, X_{t-}, y)) \right| > \frac{\epsilon}{2} \right\}} N_\theta(dt, dy) \\
&\leq \frac{4}{\epsilon^2} \int_0^T \int_{\mathbb{R}_0^d} \Big(u^\intercal \varphi_T(\theta) \nabla_\theta \ln\left(\Psi(\theta, X_{t-}, y)\right) \Big)^2 \mathbf{1}_{\left\{ \left| u^\intercal \varphi_T(\theta) \nabla_\theta \ln(\Psi(\theta, X_{t-}, y)) \right| > \frac{\epsilon}{2} \right\}} N_\theta(dt, dy) \\
&\qquad + \frac{4}{\epsilon^2} \int_0^T \int_{\mathbb{R}_0^d} \left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 - u^\intercal \varphi_T(\theta) \nabla_\theta \ln\left(\Psi(\theta, X_{t-}, y)\right) \right)^2 \\
&\qquad\qquad \times \mathbf{1}_{\left\{ \left| \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 - u^\intercal \varphi_T(\theta) \nabla_\theta \ln(\Psi(\theta, X_{t-}, y)) \right| > \frac{\epsilon}{2} \right\}} N_\theta(dt, dy).
\end{aligned}$$

Then, using hypotheses **(A6)** and **(A8)** we obtain (2.9), and hence (2.8).

Now, since $|f(x)| \leq 2|x|^3$ if $|x| \leq \frac{1}{2}$, we have for every $0 < \epsilon \leq \frac{1}{2}$,

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{R}_0^d} f\left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right) \mathbf{1}_{\left\{ \left| \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right| \leq \epsilon \right\}} N_\theta(dt, dy) \right| \\
&\leq 2 \left| \int_0^T \int_{\mathbb{R}_0^d} \left| \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right|^3 \mathbf{1}_{\left\{ \left| \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right| \leq \epsilon \right\}} N_\theta(dt, dy) \right| \\
&\leq 2\epsilon \left| \int_0^T \int_{\mathbb{R}_0^d} \left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right)^2 N_\theta(dt, dy) \right|.
\end{aligned}$$

Thus, from hypothesis **(A4)** and (2.7), we conclude that for every $\epsilon > 0$, as $T \to \infty$,

$$\int_0^T \int_{\mathbb{R}_0^d} f\left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right) \mathbf{1}_{\left\{ \left| \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right| \leq \epsilon \right\}} N_\theta(dt, dy) \xrightarrow{\mathrm{P}_\theta} 0,$$

uniformly in $\theta \in \Theta$, which finishes the desired proof. □

We end this section by recalling an important consequence of the asymptotic normality of the score function and the LAMN property, which is the conditional convolution theorem.

First, recall that a family of estimators $(\tilde{\theta}_T)_{T \geq 0}$ of the parameter θ is called regular at θ if for any $u \in \mathbb{R}^k$, as $T \to \infty$,

$$\varphi_T^{-1}(\theta) \left(\tilde{\theta}_T - (\theta + \varphi_T(\theta) u) \right) \xrightarrow{\mathcal{L}(\mathrm{P}_{\theta + \varphi_T(\theta) u})} V(\theta),$$

for some $\mathbb{R}^k$-valued random variable $V(\theta)$.

Note that taking $u = 0$, this implies that as $T \to \infty$,

$$\varphi_T^{-1}(\theta)\left(\tilde{\theta}_T - \theta\right) \overset{\mathcal{L}(\mathrm{P}_\theta)}{\longrightarrow} V(\theta).$$

The conditional convolution theorem says that when the asymptotic normality of the score function and the LAMN property hold, the asymptotic distribution of any family of regular estimators of the parameter θ is characterized by a conditional convolution between a Gaussian law and some others laws. More precisely,

Theorem 2.7 (Conditional convolution theorem)**.** [7, Theorem 9.1] *Suppose that the family of probability measures* $(\mathrm{P}_\theta^T)_{\theta\in\Theta}$ *satisfies the LAMN property at a point* θ*. Let* $(\tilde{\theta}_T)_{T\geq 0}$ *be a regular family of estimators of the parameter* θ*. Then the law of* $V(\theta)$ *conditionally on* $\Gamma(\theta)$ *is a convolution between* $\mathcal{N}\left(0, \Gamma(\theta)^{-1}\right)$ *and some other law* $G_{\Gamma(\theta)}$ *on* $\mathbb{R}^k$*, that is,*

$$\mathcal{L}\left(V(\theta)|\Gamma(\theta)\right) = \mathcal{N}\left(0, \Gamma(\theta)^{-1}\right) \star G_{\Gamma(\theta)},$$

where $G_{\Gamma(\theta)}$ *is the limiting distribution law under* P_θ *of the difference*

$$\varphi_T^{-1}(\theta)\left(\tilde{\theta}_T - \theta\right) - \Gamma(\theta)^{-1}\varphi_T(\theta)\nabla_\theta \ell_T(\theta),$$

as $T \to \infty$*, that is,*

$$G_{\Gamma(\theta)} = V(\theta) - \mathcal{N}\left(0, \Gamma(\theta)^{-1}\right).$$

This theorem yields the following definition. A family of estimators $(\tilde{\theta}_T)_{T\geq 0}$ of the parameter θ is called asymptotically efficient if as $T \to \infty$,

$$\varphi_T^{-1}(\theta)\left(\tilde{\theta}_T - \theta\right) \overset{\mathcal{L}(\mathrm{P}_\theta)}{\longrightarrow} \Gamma(\theta)^{-1/2}\mathcal{N}(0, I_k),$$

where $\mathcal{N}(0, I_k)$ and $\Gamma(\theta)$ are independent.

For instance, a family of estimators $(\tilde{\theta}_T)_{T\geq 0}$ of θ which satisfy

$$\varphi_T^{-1}(\theta)\left(\tilde{\theta}_T - \theta\right) = \varphi_T^{-1}(\theta)\left\langle \nabla_\theta \ell(\theta)\right\rangle_T^{-1} \nabla_\theta \ell_T(\theta) + o_{\mathrm{P}_\theta}(1)$$

are regular and asymptotically efficient at θ.

3. Girsanov's Theorem

This section is devoted to prove Girsanov's theorem for the diffusion process with jumps (1.1), which is needed in the proof of Theorems 2.2 and 2.3. Let us recall that in [21], Sørensen deals with the same diffusion process with jumps (1.1) but where the dimension of space on which the jumps of the Poisson random measure $p_\theta(dt, dz)$ are defined can be different from that of X^θ. The author gives sufficient conditions for the equivalence of all probability measures and then derives a complicated expression of the Radon-Nikodym derivative (see [21, Theorem 2.1]). For the proof of this result, the author reduces to check the validity of the conditions of Girsanov's theorem for semimartingales established by Jacod and Mémin (see [8, Theorem 4.2 and 4.5(b)]). In our context, we obtain an explicit expression of the Radon-Nikodym derivative with a direct proof of Girsanov's theorem.

Theorem 3.1 (Girsanov's theorem). *Assume condition* **(A1)**. *Then for all $\theta, \theta_0 \in \Theta$, the probability measures P_θ^T and $\mathrm{P}_{\theta_0}^T$ are equivalent. Furthermore, their Radon-Nikodym derivative is given by*

$$\begin{aligned}
&\frac{d\mathrm{P}_{\theta_0}^T}{d\mathrm{P}_\theta^T}\\
&= \exp\bigg\{ \int_0^T \sigma^{-1}(X_t)\left(b(\theta_0, X_t) - b(\theta, X_t)\right) \cdot dB_t - \frac{1}{2}\int_0^T |\sigma^{-1}(X_t)\left(b(\theta_0, X_t) - b(\theta, X_t)\right)|^2 dt \\
&\qquad + \int_0^T \int_{\mathbb{R}_0^d} \ln \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} N_\theta(dt, dy) - \int_0^T \int_{\mathbb{R}_0^d} \Big(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1\Big)\mu_\theta(dy)dt \bigg\}.
\end{aligned}$$

Proof. On the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathrm{P})$, we denote by $X^T = \{X_t, 0 \leq t \leq T\}$ and $Y^T = \{Y_t, 0 \leq t \leq T\}$ the solutions to equation (1.1) with parameters $\theta \in \Theta$ and $\theta_0 \in \Theta$, respectively. Let $\mathrm{P}_\theta^T, \mathrm{P}_{\theta_0}^T$ be the probability measures on the space $(D[0,T], \mathcal{B}[0,T])$ generated by the two processes, respectively.

Recall that each equation can be written as

$$X_t = X_0 + \int_0^t b(\theta, X_s)ds + \int_0^t \sigma(X_s)dB_s + \int_0^t \int_{\mathbb{R}_0^d} yN_\theta(ds, dy), \tag{3.1}$$

where $N_\theta(dt, dy)$ is a Poisson random measure with intensity $\mu_\theta(dy)dt = \Psi(\theta, X_{t-}, y)dydt$, and

$$Y_t = Y_0 + \int_0^t b(\theta_0, Y_s)ds + \int_0^t \sigma(Y_s)dB_s + \int_0^t \int_{\mathbb{R}_0^d} yN'_{\theta_0}(ds, dy), \tag{3.2}$$

where $N'_{\theta_0}(dt, dy)$ is a Poisson random measure with intensity $\mu'_{\theta_0}(dy)dt = \Psi(\theta_0, Y_{t-}, y)dydt$.

For $t \in [0,T]$, set

$$\begin{aligned}
L_t = \exp\bigg\{ &\int_0^t \sigma^{-1}(X_s)\left(b(\theta_0, X_s) - b(\theta, X_s)\right) \cdot dB_s - \frac{1}{2}\int_0^t |\sigma^{-1}(X_s)\left(b(\theta_0, X_s) - b(\theta, X_s)\right)|^2 ds \\
&+ \int_0^t \int_{\mathbb{R}_0^d} \ln \frac{\Psi(\theta_0, X_{s-}, y)}{\Psi(\theta, X_{s-}, y)} N_\theta(ds, dy) - \int_0^t \int_{\mathbb{R}_0^d} \Big(\frac{\Psi(\theta_0, X_{s-}, y)}{\Psi(\theta, X_{s-}, y)} - 1\Big)\mu_\theta(dy)ds \bigg\}.
\end{aligned}$$

By Itô's formula,

$$\begin{aligned}
L_t = 1 &+ \int_0^t L_s \sigma^{-1}(X_s)\left(b(\theta_0, X_s) - b(\theta, X_s)\right) \cdot dB_s \\
&+ \int_0^t \int_{\mathbb{R}_0^d} L_{s-}\left(\frac{\Psi(\theta_0, X_{s-}, y)}{\Psi(\theta, X_{s-}, y)} - 1\right)\left(N_\theta(ds, dy) - \mu_\theta(dy)ds\right).
\end{aligned}$$

Hence, $(L_t, 0 \leq t \leq T)$ is a non-negative $(\mathcal{F}_t, 0 \leq t \leq T)$-martingale under P and $\mathrm{E}_{\mathrm{P}}(L_t) = 1$ for all $t \in [0,T]$. On the space $(\Omega, \mathcal{F})$, we consider the probability measure Q defined as

$$dQ(\omega) = L_T(\omega)d\mathrm{P}(\omega), \quad \text{for all } \omega \in \Omega,$$

and the process $W = (W_t, 0 \leq t \leq T)$ given by

$$W_t = B_t - \int_0^t \sigma^{-1}(X_s)\left(b(\theta_0, X_s) - b(\theta, X_s)\right)ds.$$

Then, by the classical Girsanov's theorem, W is an $(\mathcal{F}_t, 0 \leq t \leq T)$-Brownian motion under Q.

Recall that under P, the compensator of $N_\theta(dt, dy)$ is given by $\mu_\theta(dy)dt = \Psi(\theta, X_{t-}, y)dydt$. We are going to show that under Q, the compensator of $N_\theta(dt, dy)$ has a representation of the form

$$\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)}\mu_\theta(dy)dt = \mu_{\theta_0}(dy)dt.$$

For this, it suffices to prove that the process $H = (H_t, 0 \leq t \leq T)$ defined as

$$H_t = \int_0^t \int_{\mathbb{R}_0^d} K(s, y) \left(N_\theta(ds, dy) - \mu_{\theta_0}(dy)ds \right)$$

is a martingale under Q, where the mapping $K : [0, T] \times \mathbb{R}_0^d \times \Omega \to \mathbb{R}^d$ is predictable and satisfies that

$$\mathrm{E}_Q \left[\int_0^T \int_{\mathbb{R}_0^d} |K(t, y)|^2 \mu_{\theta_0}(dy)dt \right] < +\infty.$$

By Itô's formula, it yields that, under P,

$$\begin{aligned}
d(L_t H_t) &= L_t dH_t + H_t dL_t + d[L, H]_t \\
&= H_t L_t \sigma^{-1}(X_t) \left(b(\theta_0, X_t) - b(\theta, X_t) \right) \cdot dB_t \\
&\quad + \int_{\mathbb{R}_0^d} L_{t-} \left(\frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} - 1 \right) \left(N_\theta(dt, dy) - \mu_\theta(dy)dt \right) \\
&\quad + \int_{\mathbb{R}_0^d} L_{t-} K(t, y) \frac{\Psi(\theta_0, X_{t-}, y)}{\Psi(\theta, X_{t-}, y)} \left(N_\theta(dt, dy) - \mu_\theta(dy)dt \right).
\end{aligned}$$

Hence, the product process $(L_t H_t, 0 \leq t \leq T)$ is an $(\mathcal{F}_t, 0 \leq t \leq T)$-martingale under P. Thus, from Bayes's formula, we obtain that $H = (H_t, 0 \leq t \leq T)$ is an $(\mathcal{F}_t, 0 \leq t \leq T)$-martingale under Q.

Now, under Q, equation (3.1) writes as

$$X_t = X_0 + \int_0^t b(\theta_0, X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \int_{\mathbb{R}_0^d} y N_{\theta_0}(ds, dy), \tag{3.3}$$

where $N_{\theta_0}(dt, dy)$ is a Poisson random measure whose intensity measure is given by $\mu_{\theta_0}(dy)dt = \Psi(\theta_0, X_{t-}, y)dydt$. Therefore, (X, W, N_{θ_0}, Q) is a weak solution to (3.2) with initial condition $X_0 = Y_0$. By the weak uniqueness of solutions to stochastic differential equations with jumps, the law of X under Q coincides with the law of Y under P. Consequently, we conclude that $\mathrm{P}_{\theta_0}^T$ is absolutely continuous with respect to P_θ^T. Proceeding in the same way, we also obtain that P_θ^T is absolutely continuous with respect to $\mathrm{P}_{\theta_0}^T$, which implies that P_θ^T and $\mathrm{P}_{\theta_0}^T$ are equivalent and their Radon-Nikodym derivative is given by L_T.

This completes the proof of the theorem. □

4. Central Limit theorem for a pure jump martingale

The aim of this section is to prove a Central Limit theorem for a stochastic integral with respect to the compensated Poisson random measure $N_\theta(dt, dy) - \mu_\theta(dy)dt$ in our context of finite intensity measure. This result is an extension of the multidimensional Central Limit

theorem for stochastic integrals with respect to a Brownian motion (see [13, Proposition 1.21]), and with respect to a process of Poisson type (see [12, Theorem 4.5.4.]). As mentioned before, this result is needed in the proof of Theorems 2.2 and 2.3. Recall that a Central Limit theorem for multivariate martingales was established by Sørensen in [21, Theorem A.1], as an extension of the Central Limit theorem for one-dimensional martingales [3, Theorem 2] by Feigin. As explained in Remark 2.4, we are assuming condition **(A6)** instead of hypothesis **1** in [21, Theorem A.1] or **(J.1)** in [15, Theorem 1], which in particular implies **(J.3)** in [15, Theorem 1].

Theorem 4.1. *Assume hypotheses* **(A1)**, **(A4)**, (2.3) *and* **(A6)**, *and that the function* $\Psi(\theta, x, y)$ *is differentiable with respect to* θ. *Consider the square integrable martingale*

$$M_T(\theta) = \int_0^T \int_{\mathbb{R}_0^d} u^\mathsf{T} \varphi_T(\theta) \nabla_\theta \ln\left(\Psi(\theta, X_{t-}, y)\right) \left(N_\theta(dt, dy) - \mu_\theta(dy)dt\right).$$

Then, for all $u \in \mathbb{R}^k$, *as* $T \to \infty$,

$$M_T(\theta) \overset{\mathcal{L}(\mathrm{P}_\theta)}{\longrightarrow} u^\mathsf{T} \Gamma_2(\theta)^{1/2} \mathcal{N}(0, I_k),$$

uniformly in $\theta \in \Theta$, *where* $\mathcal{N}(0, I_k)$ *is a centered* $\mathbb{R}^k$*-valued Gaussian variable independent of* $\Gamma_2(\theta)$, *the matrix from condition* **(A4)**.

Proof. Consider a Poisson random measure $\widetilde{N}_\theta(dt, dy)$ on $[0, T+1] \times \mathbb{R}_0^d$ with intensity measure $\widetilde{\mu}_\theta(dy)dt$ defined as

$$\widetilde{\mu}_\theta(dy)dt = \begin{cases} \mu_\theta(dy)dt, & 0 \leq t \leq T, \\ \dfrac{1}{\Pi_{i=1}^k \varphi_{i,T}^2(\theta)} \mu_\theta(dy)dt, & T < t \leq T+1, \end{cases}$$

where $\varphi_{i,T}(\theta)$ denote the entries of the diagonal matrix $\varphi_T(\theta)$.

We also consider the predictable process $h : \Theta \times [0, T+1] \times \mathbb{R}_0^d \times \Omega \to \mathbb{R}$ given by

$$h(\theta, t, y) = \begin{cases} u^\mathsf{T} \varphi_T(\theta) \nabla_\theta \ln\left(\Psi(\theta, X_{t-}, y)\right), & 0 \leq t \leq T, \\ \dfrac{\Pi_{i=1}^k \varphi_{i,T}(\theta)}{\sqrt{\lambda(\theta)}} \left| u^\mathsf{T} \Gamma_2(\theta) u \right|^{1/2}, & T < t \leq T+1, \end{cases}$$

where recall that $\lambda(\theta) = \int_{\mathbb{R}_0^d} \mu_\theta(dy) < \infty$, P_θ-a.s.

We next introduce the stopping time

$$\tau_T(\theta) = \inf\Big\{t \geq 0 : \int_0^t \int_{\mathbb{R}_0^d} |h(\theta, s, y)|^2 \, \widetilde{\mu}_\theta(dy)ds \geq u^\mathsf{T} \Gamma_2(\theta) u\Big\}.$$

Then, for all $\theta \in \Theta$, $\tau_T(\theta) \in [0, T+1]$, P_θ-a.s., since

$$\begin{aligned} \int_0^{T+1} \int_{\mathbb{R}_0^d} |h(\theta, s, y)|^2 \, \widetilde{\mu}_\theta(dy)ds &= \int_0^T \int_{\mathbb{R}_0^d} \left| u^\mathsf{T} \varphi_T(\theta) \nabla_\theta \ln\left(\Psi(\theta, X_{s-}, y)\right) \right|^2 \mu_\theta(dy)ds \\ &\quad + \int_T^{T+1} \int_{\mathbb{R}_0^d} \frac{\Pi_{i=1}^k \varphi_{i,T}^2(\theta)}{\lambda(\theta)} u^\mathsf{T} \Gamma_2(\theta) u \frac{1}{\Pi_{i=1}^k \varphi_{i,T}^2(\theta)} \mu_\theta(dy)ds \\ &\geq u^\mathsf{T} \Gamma_2(\theta) u. \end{aligned}$$

Consider now the stopped stochastic integral given by

$$I(\tau_T(\theta)) = \int_0^{\tau_T(\theta)} \int_{\mathbb{R}_0^d} h(\theta,s,y) \left(\widetilde{N}_\theta(ds,dy) - \widetilde{\mu}_\theta(dy)ds \right).$$

For $t \in [0, T+1]$ and $v \in \mathbb{R}$, we introduce the stochastic process

$$U_T(t) = \exp\left\{ iv \int_0^t \int_{\mathbb{R}_0^d} h(\theta,s,y) \left(\widetilde{N}_\theta(ds,dy) - \widetilde{\mu}_\theta(dy)ds \right) + \frac{v^2}{2} \int_0^t \int_{\mathbb{R}_0^d} |h(\theta,s,y)|^2 \widetilde{\mu}_\theta(dy)ds \right\}.$$

Then, by Itô's formula

$$\begin{aligned} U_T(t) &= 1 + \int_0^t \int_{\mathbb{R}_0^d} U_T(s-) \left(e^{ivh(\theta,s,y)} - 1 \right) \left(\widetilde{N}_\theta(ds,dy) - \widetilde{\mu}_\theta(dy)ds \right) \\ &\quad + \int_0^t \int_{\mathbb{R}_0^d} U_T(s) \left(e^{ivh(\theta,s,y)} - 1 - ivh(\theta,s,y) + \frac{v^2}{2} |h(\theta,s,y)|^2 \right) \widetilde{\mu}_\theta(dy)ds. \end{aligned}$$

Now, from the definitions of $\tau := \tau_T(\theta)$ and $U_T(\tau)$, it yields that

$$\exp\left\{ iv \int_0^\tau \int_{\mathbb{R}_0^d} h(\theta,s,y) \left(\widetilde{N}_\theta(ds,dy) - \widetilde{\mu}_\theta(dy)ds \right) \right\} = U_T(\tau) e^{-\frac{v^2}{2} u^\mathsf{T} \Gamma_2(\theta) u}.$$

Taking expectation in both sides of the last equality, we get that

$$\Phi_{I(\tau)}(v) = \mathrm{E}_\theta \left[U_T(\tau) e^{-\frac{v^2}{2} u^\mathsf{T} \Gamma_2(\theta) u} \right], \tag{4.1}$$

where $\Phi_Y(v)$ denotes the characteristic function of the random variable Y at the point $v \in \mathbb{R}$.

We are now going to show that for all $v \in \mathbb{R}$,

$$\lim_{T\to\infty} \Phi_{I(\tau)}(v) = \mathrm{E}_\theta \left[e^{-\frac{v^2}{2} u^\mathsf{T} \Gamma_2(\theta) u} \right], \tag{4.2}$$

uniformly in $\theta \in \Theta$. By (4.1), it suffices to prove that for all $v \in \mathbb{R}$,

$$\lim_{T\to\infty} \left| \mathrm{E}_\theta \left[\left(U_T(\tau) - 1 \right) e^{-\frac{v^2}{2} u^\mathsf{T} \Gamma_2(\theta) u} \right] \right| = 0,$$

uniformly in $\theta \in \Theta$.

Observe that

$$\begin{aligned} &\left(U_T(\tau) - 1 \right) e^{-\frac{v^2}{2} u^\mathsf{T} \Gamma_2(\theta) u} \\ &= e^{-\frac{v^2}{2} u^\mathsf{T} \Gamma_2(\theta) u} \int_0^\tau \int_{\mathbb{R}_0^d} U_T(s-) \left(e^{ivh(\theta,s,y)} - 1 \right) \left(\widetilde{N}_\theta(ds,dy) - \widetilde{\mu}_\theta(dy)ds \right) \\ &+ e^{-\frac{v^2}{2} u^\mathsf{T} \Gamma_2(\theta) u} \int_0^\tau \int_{\mathbb{R}_0^d} U_T(s) \left(e^{ivh(\theta,s,y)} - 1 - ivh(\theta,s,y) + \frac{v^2}{2} |h(\theta,s,y)|^2 \right) \widetilde{\mu}_\theta(dy)ds. \end{aligned}$$

Taking expectations in both sides of the last equality, we get that

$$\left| \mathrm{E}_\theta \left[\left(U_T(\tau) - 1 \right) e^{-\frac{v^2}{2} u^\mathsf{T} \Gamma_2(\theta) u} \right] \right|$$

$$\leq \mathrm{E}_\theta\left[e^{-\frac{v^2}{2}u^\mathsf{T}\Gamma_2(\theta)u}\int_0^\tau\int_{\mathbb{R}_0^d}|U_T(s)|\left|e^{ivh(\theta,s,y)}-1-ivh(\theta,s,y)+\frac{v^2}{2}\left|h(\theta,s,y)\right|^2\right|\widetilde{\mu}_\theta(dy)ds\right].$$

Using the following inequalities, valid for all $x\in\mathbb{R}$,

$$|e^{ix}-1-ix|\leq\frac{1}{2}x^2,\quad \text{and } |e^{ix}-1-ix+\frac{x^2}{2}|\leq\frac{1}{6}|x|^3,$$

and the fact that $|U_T(s)|\leq e^{\frac{v^2}{2}u^\mathsf{T}\Gamma_2(\theta)u}$ for all $s\leq\tau$, we obtain that, for all $\epsilon>0$,

$$\begin{aligned}
&\left|\mathrm{E}_\theta\left[\left(U_T(\tau)-1\right)e^{-\frac{v^2}{2}u^\mathsf{T}\Gamma_2(\theta)u}\right]\right|\\
&\leq \mathrm{E}_\theta\left[\int_0^\tau\int_{\mathbb{R}_0^d}\left|e^{ivh(\theta,s,y)}-1-ivh(\theta,s,y)+\frac{v^2}{2}\left|h(\theta,s,y)\right|^2\right|\widetilde{\mu}_\theta(dy)ds\right]\\
&= \mathrm{E}_\theta\Bigg[\int_0^\tau\int_{\mathbb{R}_0^d}\left|e^{ivh(\theta,s,y)}-1-ivh(\theta,s,y)+\frac{v^2}{2}\left|h(\theta,s,y)\right|^2\right|\\
&\qquad\qquad\qquad\times\left(\mathbf{1}_{\{|vh(\theta,s,y)|\leq\epsilon\}}+\mathbf{1}_{\{|vh(\theta,s,y)|>\epsilon\}}\right)\widetilde{\mu}_\theta(dy)ds\Bigg]\\
&\leq\frac{|v|^3}{6}\mathrm{E}_\theta\left[\int_0^\tau\int_{\mathbb{R}_0^d}|h(\theta,s,y)|^3\mathbf{1}_{\{|vh(\theta,s,y)|\leq\epsilon\}}\widetilde{\mu}_\theta(dy)ds\right]\\
&\quad+v^2\mathrm{E}_\theta\left[\int_0^\tau\int_{\mathbb{R}_0^d}|h(\theta,s,y)|^2\mathbf{1}_{\{|vh(\theta,s,y)|\geq\epsilon\}}\widetilde{\mu}_\theta(dy)ds\right]\\
&\leq\frac{v^2}{6}\epsilon\mathrm{E}_\theta\left[\int_0^\tau\int_{\mathbb{R}_0^d}|h(\theta,s,y)|^2\widetilde{\mu}_\theta(dy)ds\right]\\
&+v^2\mathrm{E}_\theta\left[\int_T^{T+1}\int_{\mathbb{R}_0^d}\frac{\Pi_{i=1}^k\varphi_{i,T}^2(\theta)}{\lambda(\theta)}u^\mathsf{T}\Gamma_2(\theta)u\mathbf{1}_{\left\{\left|v\frac{\Pi_{i=1}^k\varphi_{i,T}(\theta)}{\sqrt{\lambda(\theta)}}\left|u^\mathsf{T}\Gamma_2(\theta)u\right|^{1/2}\right|\geq\epsilon\right\}}\frac{1}{\Pi_{i=1}^k\varphi_{i,T}^2(\theta)}\mu_\theta(dy)ds\right]\\
&+v^2\mathrm{E}_\theta\left[\int_0^T\int_{\mathbb{R}_0^d}\left|u^\mathsf{T}\varphi_T(\theta)\nabla_\theta\ln\left(\Psi(\theta,X_{t-},y)\right)\right|^2\mathbf{1}_{\left\{\left|vu^\mathsf{T}\varphi_T(\theta)\nabla_\theta\ln(\Psi(\theta,X_{t-},y))\right|\geq\epsilon\right\}}\mu_\theta(dy)ds\right]\\
&\leq\frac{v^2}{6}\epsilon\mathrm{E}_\theta\left[u^\mathsf{T}\Gamma_2(\theta)u\right]+v^2\mathrm{E}_\theta\left[u^\mathsf{T}\Gamma_2(\theta)u\mathbf{1}_{\left\{\frac{\Pi_{i=1}^k\varphi_{i,T}(\theta)}{\sqrt{\lambda(\theta)}}\left|u^\mathsf{T}\Gamma_2(\theta)u\right|^{1/2}\geq\epsilon|v|^{-1}\right\}}\right]\\
&+v^2\int_0^T\int_{\mathbb{R}_0^d}\mathrm{E}_\theta\left[\left|u^\mathsf{T}\varphi_T(\theta)\nabla_\theta\ln\left(\Psi(\theta,X_{t-},y)\right)\right|^2\mathbf{1}_{\left\{\left|u^\mathsf{T}\varphi_T(\theta)\nabla_\theta\ln(\Psi(\theta,X_{t-},y))\right|>\epsilon\right\}}\Psi(\theta,X_{s-},y)\right]dyds.
\end{aligned}$$

Hence, hypothesis **(A6)** and the fact that $\varphi_{i,T}(\theta)$ tends to zero as $T\to\infty$, conclude the proof of (4.2).

We next consider the events $A=\{\omega:\tau(\omega)<T\}$ and $A^c=\{\omega:\tau(\omega)\in(T,T+1]\}$. Then, for any $\delta>0$ and $\epsilon>0$, by [1, Lemma 4.2.8] and the definition of τ, we have that

$$\sup_{\theta\in\Theta}\mathrm{P}_\theta\left\{|M_T(\theta)-I(\tau)|>\delta\right\}$$

$$
\begin{aligned}
&= \sup_{\theta\in\Theta} \mathrm{P}_\theta \left\{ \left| \int_0^{T+1} \int_{\mathbb{R}_0^d} h(\theta,t,y) \left(\mathbf{1}_{\{t\leq T\}} - \mathbf{1}_{\{t\leq \tau\}}\right) \left(\widetilde{N}_\theta(dt,dy) - \widetilde{\mu}_\theta(dy)dt\right) \right| > \delta \right\} \\
&\leq \frac{\epsilon}{\delta^2} + \sup_{\theta\in\Theta} \mathrm{P}_\theta \left\{ \int_0^{T+1} \int_{\mathbb{R}_0^d} |h(\theta,t,y)|^2 \, |\mathbf{1}_{\{t\leq T\}} - \mathbf{1}_{\{t\leq \tau\}}| \widetilde{\mu}_\theta(dy)dt > \epsilon \right\} \\
&= \frac{\epsilon}{\delta^2} + \sup_{\theta\in\Theta} \mathrm{P}_\theta \left\{ \int_0^{T+1} \int_{\mathbb{R}_0^d} |h(\theta,t,y)|^2 \, (\mathbf{1}_{\{t\leq T\}} - \mathbf{1}_{\{t\leq \tau\}}) \widetilde{\mu}_\theta(dy)dt > \epsilon, A \right\} \\
&\quad + \sup_{\theta\in\Theta} \mathrm{P}_\theta \left\{ \int_0^{T+1} \int_{\mathbb{R}_0^d} |h(\theta,t,y)|^2 \, (\mathbf{1}_{\{t\leq T\}} - \mathbf{1}_{\{t\leq \tau\}}) \widetilde{\mu}_\theta(dy)dt > \epsilon, \tau = T \right\} \\
&\quad + \sup_{\theta\in\Theta} \mathrm{P}_\theta \left\{ \int_0^{T+1} \int_{\mathbb{R}_0^d} |h(\theta,t,y)|^2 \, (\mathbf{1}_{\{t\leq \tau\}} - \mathbf{1}_{\{t\leq T\}}) \widetilde{\mu}_\theta(dy)dt > \epsilon, A^c \right\} \\
&= \frac{\epsilon}{\delta^2} + \sup_{\theta\in\Theta} \mathrm{P}_\theta \left\{ \langle M(\theta)\rangle_T - u^\mathsf{T}\Gamma_2(\theta)u > \epsilon, A \right\} + \sup_{\theta\in\Theta} \mathrm{P}_\theta \left\{ u^\mathsf{T}\Gamma_2(\theta)u - \langle M(\theta)\rangle_T > \epsilon, A^c \right\} \\
&\leq \frac{\epsilon}{\delta^2} + \sup_{\theta\in\Theta} \mathrm{P}_\theta \left\{ \left| \langle M(\theta)\rangle_T - u^\mathsf{T}\Gamma_2(\theta)u \right| > \epsilon \right\}.
\end{aligned}
$$

Therefore, choosing $\epsilon = \delta^3$ and using hypothesis **(A4)**, we get that

$$
\sup_{\theta\in\Theta} \mathrm{P}_\theta \left\{ |M_T(\theta) - I(\tau)| > \delta \right\} \leq \delta + \sup_{\theta\in\Theta} \mathrm{P}_\theta \left\{ \left| \langle M(\theta)\rangle_T - u^\mathsf{T}\Gamma_2(\theta)u \right| > \delta^3 \right\} \leq 2\delta,
$$

which shows that

$$
\mathrm{P}_\theta - \lim_{T\to\infty} |M_T(\theta) - I(\tau)| = 0, \tag{4.3}
$$

uniformly in $\theta \in \Theta$.

Finally, (4.2) and (4.3) imply that for all $v \in \mathbb{R}$,

$$
\lim_{T\to\infty} \Phi_{M_T(\theta)}(v) = \mathrm{E}_\theta \left[e^{-\frac{v^2}{2} u^\mathsf{T}\Gamma_2(\theta)u} \right],
$$

uniformly in $\theta \in \Theta$, which concludes the desired proof. □

5. LAN property for ergodic diffusion processes with jumps

In this section, we seek sufficient conditions in order for the LAN property to hold when the diffusion process with jumps X^θ (1.1) is ergodic, as a consequence of Theorem 2.3.

Let $X^\theta = \{X_t\}_{t\geq 0}$ be the solution to equation (1.1), that is,

$$
X_t = X_0 + \int_0^t a(\theta, X_s)ds + \int_0^t \sigma(X_s)dB_s + \int_0^t \int_{\mathbb{R}_0^d} c(\theta, X_{s-}, z)(p_\theta(ds,dz) - \nu_\theta(dz)ds).
$$

Recall that we have rewritten this equation as

$$
X_t = X_0 + \int_0^t b(\theta, X_s)ds + \int_0^t \sigma(X_s)dB_s + \int_0^t \int_{\mathbb{R}_0^d} yN_\theta(dt,dy),
$$

where

$$
b(\theta, X_t) = a(\theta, X_t) - \int_{\mathbb{R}_0^d} y\mu_\theta(dy),
$$

and $N_\theta(dt, dy)$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_0^d$ with intensity measure $\mu_\theta(dy)dt = \Psi(\theta, X_{t-}, y)dydt$.

As is well-known, X^θ is a homogeneous Markov process (see [1, Theorem 6.4.6]). Let us introduce the ergodic assumption.

(C1) The process X^θ is ergodic, that is, there exists an invariant probability measure $\pi_\theta(dx)$ such that as $T \to \infty$,

$$\frac{1}{T}\int_0^T g(\theta, X_t)dt \xrightarrow{\mathrm{P}_\theta} \int_{\mathbb{R}^d} g(\theta, x)\pi_\theta(dx),$$

for any π_θ-integrable function g, uniformly in $\theta \in \Theta$.

Several examples of ergodic diffusion processes with jumps are given in [16], [17], and [23]. Moreover, results on ergodicity and exponential ergodicity of diffusion processes with jumps have been established by Masuda in [16, 17]. However, in these papers ergodicity and exponentially ergodicity are understood in the sense of [18], which both are stronger than the ergodicity in the sense **(C1)**.

We next show the following consequence of assuming that X^θ is ergodic.

Lemma 5.1. *If assumptions* **(A1)**, **(A2)**, *and* **(C1)** *are satisfied, then assumptions* **(A3)** *and* **(A4)** *hold with $\varphi_T(\theta)$ the diagonal matrix with entries equal to $\frac{1}{\sqrt{T}}$, and*

$$\Gamma_1(\theta) = \int_{\mathbb{R}^d} \left(\nabla_\theta b(\theta, x)\right)^\mathsf{T} (\sigma^{-1}(x))^\mathsf{T} \sigma^{-1}(x)\nabla_\theta b(\theta, x)\pi_\theta(dx),$$

$$\Gamma_2(\theta) = \int_{\mathbb{R}^d}\int_{\mathbb{R}_0^d} \nabla_\theta \ln\Psi(\theta, x, y)\left(\nabla_\theta \ln\Psi(\theta, x, y)\right)^\mathsf{T} \mu_\theta(dy)\pi_\theta(dx).$$

Proof. By ergodicity, as $T \to \infty$,

$$\frac{1}{T}\int_0^T \left(\nabla_\theta b(\theta, X_t)\right)^\mathsf{T} (\sigma^{-1}(X_t))^\mathsf{T} \sigma^{-1}(X_t)\nabla_\theta b(\theta, X_t)dt \xrightarrow{\mathrm{P}_\theta} \Gamma_1(\theta),$$

$$\frac{1}{T}\int_0^T \int_{\mathbb{R}_0^d} \nabla_\theta \ln\Psi(\theta, X_{t-}, y)\left(\nabla_\theta \ln\Psi(\theta, X_{t-}, y)\right)^\mathsf{T} \mu_\theta(dy)dt \xrightarrow{\mathrm{P}_\theta} \Gamma_2(\theta),$$

which implies the desired result. □

As a consequence of this result, we have the following immediate consequence of Theorem 2.2.

Theorem 5.2. *Suppose that conditions* **(A1)**, **(A2)**, **(A5)**, *and* **(C1)** *are satisfied, and* **(A6)** *holds with $\varphi_T(\theta)$ the diagonal matrix with entries equal to $\frac{1}{\sqrt{T}}$. Then the score function is asymptotically normal uniformly for all $\theta \in \Theta$ with asymptotic Fisher information matrix $\Gamma(\theta)$. That is, as $T \to \infty$,*

$$\frac{1}{\sqrt{T}}\nabla_\theta \ell_T(\theta) \xrightarrow{\mathcal{L}(\mathrm{P}_\theta)} \mathcal{N}\big(0, \Gamma(\theta)\big),$$

uniformly in $\theta \in \Theta$, where $\mathcal{N}\big(0, \Gamma(\theta)\big)$ is a centered $\mathbb{R}^k$-valued Gaussian variable with covariance matrix $\Gamma(\theta)$.

We next derive the LAN property. For this, we need the following additional assumptions.

(C2) For all $u \in \mathbb{R}^k$, as $T \to \infty$,

$$\int_{\mathbb{R}^d} \left| \sigma^{-1}(x) \left(b\left(\theta + \frac{u}{\sqrt{T}}, x\right) - b(\theta, x) - \nabla_\theta b(\theta, x) \frac{u}{\sqrt{T}} \right) \right|^2 \pi_\theta(dx) = o\left(\frac{1}{T}\right),$$

uniformly in $\theta \in \Theta$.

(C3) For all $u \in \mathbb{R}^k$, as $T \to \infty$,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}_0^d} \left(\frac{\Psi(\theta + \frac{u}{\sqrt{T}}, x, y)}{\Psi(\theta, x, y)} - 1 - \frac{u^\mathsf{T}}{\sqrt{T}} \nabla_\theta \ln\left(\Psi(\theta, x, y)\right) \right)^2 \mu_\theta(dy) \pi_\theta(dx) = o\left(\frac{1}{T}\right),$$

uniformly in $\theta \in \Theta$.

We next state the main result of this section.

Theorem 5.3. *Suppose that conditions* **(A1)**, **(A2)**, **(A5)**, **(C1)**, **(C2)** *and* **(C3)** *are fulfilled, and* **(A6)** *holds with $\varphi_T(\theta)$ the diagonal matrix with entries equal to $\frac{1}{\sqrt{T}}$. Then the LAN property holds for all $\theta \in \Theta$ with rate of convergence $\sqrt{T}$ and asymptotic Fisher information matrix $\Gamma(\theta)$, that is, for all $u \in \mathbb{R}^k$, as $T \to \infty$,*

$$\log \frac{d\mathrm{P}^T_{\theta + \frac{u}{\sqrt{T}}}}{d\mathrm{P}^T_\theta} \xrightarrow{\mathcal{L}(\mathrm{P}_\theta)} u^\mathsf{T} \mathcal{N}\big(0, \Gamma(\theta)\big) - \frac{1}{2} u^\mathsf{T} \Gamma(\theta) u,$$

where $\mathcal{N}\big(0, \Gamma(\theta)\big)$ is a centered $\mathbb{R}^k$-valued Gaussian variable with covariance matrix $\Gamma(\theta)$.

Proof. By ergodicity, as $T \to \infty$,

$$\begin{aligned} &\frac{1}{T} \int_0^T \left| \sigma^{-1}(X_t) \left(b(\theta + \frac{u}{\sqrt{T}}, X_t) - b(\theta, X_t) - \nabla_\theta b(\theta, X_t) \frac{u}{\sqrt{T}} \right) \right|^2 dt \\ &\xrightarrow{\mathrm{P}_\theta} \int_{\mathbb{R}^d} \left| \sigma^{-1}(x) \left(b(\theta + \frac{u}{\sqrt{T}}, x) - b(\theta, x) - \nabla_\theta b(\theta, x) \frac{u}{\sqrt{T}} \right) \right|^2 \pi_\theta(dx), \end{aligned}$$

which, together with **(C2)** gives **(A7)**.

Again by ergodicity, as $T \to \infty$,

$$\begin{aligned} &\frac{1}{T} \int_0^T \int_{\mathbb{R}_0^d} \left| \frac{\Psi(\theta + \frac{u}{\sqrt{T}}, X_t, y)}{\Psi(\theta, X_t, y)} - 1 - \frac{1}{\sqrt{T}} u^\mathsf{T} \nabla_\theta \ln\left(\Psi(\theta, X_t, y)\right) \right|^2 \mu_\theta(dy) dt \\ &\xrightarrow{\mathrm{P}_\theta} \int_{\mathbb{R}^d} \int_{\mathbb{R}_0^d} \left| \frac{\Psi(\theta + \frac{u}{\sqrt{T}}, x, y)}{\Psi(\theta, x, y)} - 1 - \frac{1}{\sqrt{T}} u^\mathsf{T} \nabla_\theta \ln\left(\Psi(\theta, x, y)\right) \right|^2 \mu_\theta(dy) \pi_\theta(dx), \end{aligned}$$

which, together with **(C3)** gives **(A8)**.

Then, the desired LAN property follows from Lemma 5.1 and Theorem 2.3. □

As a consequence of the LAN property, an asymptotic lower bound for the variance of any family of unbiased estimators can be obtained. More precisely,

Theorem 5.4 (Minimax theorem)**.** [7, Theorem 12.1] *Suppose that the family of probability measures* $(\mathrm{P}_\theta^T)_{\theta\in\Theta}$ *satisfies the LAN property at a point* θ*. Let* $(\tilde{\theta}_T)_{T\geq 0}$ *be a family of estimators of the parameter* θ *and* $l : \mathbb{R}^k \to [0, +\infty)$ *be a loss function of the form* $l(0) = 0, l(x) = l(|x|)$ *and* $l(|x|) \leq l(|y|)$ *if* $|x| \leq |y|$*. Then*

$$\lim_{\delta\to 0} \liminf_{T\to\infty} \sup_{|\theta'-\theta|<\delta} \mathrm{E}_{\theta'} \left[l \left(\varphi_T^{-1}(\theta) \left(\tilde{\theta}_T - \theta' \right) \right) \right] \geq \mathrm{E}_\theta \left[l \left(Z \right) \right],$$

where $\mathcal{L}(Z) = \mathcal{N}(0, \Gamma(\theta)^{-1})$*.*

In particular, when we take the quadratic loss function $l(u) = |u|^2$, the above inequality gives an asymptotic lower bound for the covariance matrix of any family of unbiased estimators, which is given by $\Gamma(\theta)^{-1}$.

6. Examples

In this section we consider the following particular case of equation (1.2), which is the following

$$X_t = X_0 + \theta_1 \int_0^t X_s ds + \sigma B_t + \int_0^t \int_{\mathbb{R}_0} y N_{\theta_2}(ds, dy), \tag{6.1}$$

where X_0 is a random variable with finite second moment, $\sigma \geq 0$, $\theta = (\theta_1, \theta_2) \in \Theta = \mathbb{R} \times \tilde{\Theta}$ where $\tilde{\Theta}$ is a open subset of $\mathbb{R}^{k-1}$, for some integer $k > 1$, $N_{\theta_2}(ds, dy)$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}_0$ with intensity measure $\mu_{\theta_2}(dy)dt = f(\theta_2, y)dydt$, where $f : \Theta \times \mathbb{R} \to \mathbb{R}_+$ is a Borel function such that $f(\theta_2, \{0\}) = 0$, and $0 < \lambda(\theta_2) := \int_{\mathbb{R}_0} f(\theta_2, y)dy < \infty$, for all $\theta_2 \in \tilde{\Theta}$. We also assume that f is differentiable with respect to θ_2, and that f and $\nabla_{\theta_2} f$ are continuous in θ_2.

Let us now consider the following cases:

Case 1: $\sigma = 0$, $\theta_1 = 0$.

The term from condition **(A4)** writes as

$$T \int_{\mathbb{R}_0} \left(u^\mathsf{T} \varphi_T(\theta_2) \nabla_{\theta_2} \ln \left(f(\theta_2, y) \right) \right)^2 f(\theta_2, y) dy,$$

for $u \in \mathbb{R}^{k-1}$.

Assume the folllowing conditions on the Lévy density f.

(H1) For all $\theta_2 \in \tilde{\Theta}$,

$$\int_{\mathbb{R}_0} \frac{\nabla_{\theta_2} f(\theta_2, y)(\nabla_{\theta_2} f(\theta_2, y))^\mathsf{T}}{f(\theta_2, y)} dy < \infty.$$

(H2) For all $u \in \mathbb{R}^{k-1}$, as $T \to \infty$,

$$\int_{\mathbb{R}_0} \left| \frac{f(\theta_2 + \frac{u}{\sqrt{T}}, y)}{f(\theta_2, y)} - 1 - \frac{1}{\sqrt{T}} u^\mathsf{T} \nabla_{\theta_2} \ln \left(f(\theta_2, y) \right) \right|^2 f(\theta_2, y) dy = o \left(\frac{1}{T} \right),$$

uniformly in $\theta_2 \in \tilde{\Theta}$.

Then, assuming **(H1)** and taking $\varphi_T(\theta_2)$ as the $(k-1)\times(k-1)$ diagonal matrix with entries equal to $\frac{1}{\sqrt{T}}$, we get that conditions **(A4)** and **(A5)** hold for all $T \geq 0$ with

$$\tilde{\Gamma}_2(\theta_2) := \int_{\mathbb{R}_0} \frac{\nabla_{\theta_2} f(\theta_2, y)(\nabla_{\theta_2} f(\theta_2, y))^{\mathsf{T}}}{f(\theta_2, y)} dy, \tag{6.2}$$

which is a positive definite $(k-1)\times(k-1)$ matrix for all $\theta_2 \in \tilde{\Theta}$.

Moreover, condition **(A6)** is satisfied for all $\theta_2 \in \tilde{\Theta}$ since for all $\epsilon > 0$ and $u \in \mathbb{R}^{k-1}$, we have that as $T \to \infty$,

$$\int_{\mathbb{R}_0} \left(u^{\mathsf{T}} \nabla_{\theta_2} \ln\left(f(\theta_2, y)\right)\right)^2 \mathbf{1}_{\left\{\left|u^{\mathsf{T}} \nabla_{\theta_2} \ln(f(\theta_2, y))\right| > \epsilon\sqrt{T}\right\}} f(\theta_2, y) dy = o(1),$$

uniformly in $\theta_2 \in \tilde{\Theta}$.

The score function is given by

$$\nabla_{\theta_2} \ell_T(\theta_2) = \int_0^T \int_{\mathbb{R}_0} \nabla_{\theta_2} \ln\left(f(\theta_2, y)\right) \left(N_{\theta_2}(dt, dy) - \mu_{\theta_2}(dy)dt\right).$$

Then, under condition **(H1)**, Theorem 2.2 implies that the score function is asymptotically normal with asymptotic Fisher information $\tilde{\Gamma}_2(\theta_2)$. Moreover, assuming additionally condition **(H2)** we get that the family of probability measures $(\mathrm{P}_\theta^T)_{\theta\in\Theta}$ has the LAN property with rate of convergence $\sqrt{T}$ and asymptotic Fisher information matrix $\tilde{\Gamma}_2(\theta_2)$.

A particular case, when X is a one-dimensional Poisson process with intensity $\lambda(\theta)$, where Θ is an open subset of $\mathbb{R}$, the LAN property holds with rate of convergence $\sqrt{T}$ and asymptotic Fisher information $\Gamma(\theta) = \frac{(\lambda'(\theta))^2}{\lambda(\theta)}$.

Case 2: $\sigma > 0$, $\theta_1 > 0$.

Taking the diagonal entries of the $k \times k$ diagonal matrix $\varphi_T(\theta)$ as $\varphi_{i,T}(\theta) = \frac{1}{\sqrt{T}}$ for $i = 2, ..., k$, and assuming **(H1)**, we get that condition **(A4)** is satisfied with

$$\Gamma_2(\theta) = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{\Gamma}_2(\theta_2), \end{pmatrix}$$

where $\tilde{\Gamma}_2(\theta_2)$ denotes the $(k-1)\times(k-1)$ matrix defined in (6.2).

Recall that, by Itô's formula, the unique solution to equation (6.1) is given by

$$X_t = e^{\theta_1 t} X_0 + \sigma \int_0^t e^{\theta_1(t-s)} dB_s + \int_0^t \int_{\mathbb{R}_0} e^{\theta_1(t-s)} y N_{\theta_2}(ds, dy). \tag{6.3}$$

Suppose now that $\rho_1 := \int_{\mathbb{R}_0} y \mu_{\theta_2}(dy) < \infty$ and $\rho_2 := \int_{\mathbb{R}_0} y^2 \mu_{\theta_2}(dy) < \infty$, for all $\theta_2 \in \tilde{\Theta}$. By Itô's formula, we have that

$$\begin{aligned} X_t^2 = e^{2\theta_1 t} X_0^2 &+ 2\rho_1 \int_0^t e^{\theta_1(t-s)} X_s ds + 2\sigma \int_0^t e^{\theta_1(t-s)} X_s dB_s + \left(\sigma^2 + \rho_2\right) \int_0^t e^{2\theta_1(t-s)} ds \\ &+ \int_0^t \int_{\mathbb{R}_0} \left(2e^{\theta_1(t-s)} y X_s + e^{2\theta_1(t-s)} y^2\right) \left(N_{\theta_2}(ds, dy) - \mu_{\theta_2}(dy)ds\right), \end{aligned}$$

and notice that

$$\mathrm{E}_\theta[X_t] = e^{\theta_1 t}\left(\mathrm{E}[X_0] + \frac{\rho_1}{\theta_1}\right) - \frac{\rho_1}{\theta_1}.$$

Thus, we have that

$$\mathrm{E}_\theta[X_t^2] = e^{2\theta_1 t}\mathrm{E}[X_0^2] + 2\rho_1\left(\mathrm{E}[X_0] + \frac{\rho_1}{\theta_1}\right)e^{\theta_1 t}t + \frac{2\rho_1^2}{\theta_1^2}\left(1 - e^{\theta_1 t}\right) - \frac{\sigma^2 + \rho_2}{2\theta_1}\left(1 - e^{2\theta_1 t}\right).$$

from which we deduce that

$$\begin{aligned}\int_0^T \mathrm{E}_\theta[X_t^2]dt &= \frac{1}{2\theta_1}\left(\mathrm{E}[X_0^2] + \frac{\sigma^2 + \rho_2}{2\theta_1}\right)\left(e^{2\theta_1 T} - 1\right)\\ &\quad + \left(-\frac{2\rho_1^2}{\theta_1^3} - \frac{2\rho_1}{\theta_1^2}\left(\mathrm{E}[X_0] + \frac{\rho_1}{\theta_1}\right)\right)\left(e^{\theta_1 T} - 1\right) \qquad (6.4)\\ &\quad + \left(\frac{2\rho_1^2}{\theta_1^2} - \frac{\sigma^2 + \rho_2}{2\theta_1}\right)T + \frac{2\rho_1}{\theta_1}\left(\mathrm{E}[X_0] + \frac{\rho_1}{\theta_1}\right)Te^{\theta_1 T}.\end{aligned}$$

Next, we write

$$X_t = e^{\theta_1 t}X_0 + \rho_1\int_0^t e^{\theta_1(t-s)}ds + \sigma\int_0^t e^{\theta_1(t-s)}dB_s + \int_0^t\int_{\mathbb{R}_0} e^{\theta_1(t-s)}y\left(N_{\theta_2}(ds,dy) - \mu_{\theta_2}(dy)ds\right).$$

Now, consider the martingale

$$\begin{aligned}M_t &= e^{-\theta_1 t}X_t - X_0 + \frac{e^{-\theta_1 t} - 1}{\theta_1}\rho_1\\ &= \sigma\int_0^t e^{-\theta_1 s}dB_s + \int_0^t\int_{\mathbb{R}_0} e^{-\theta_1 s}y\left(N_{\theta_2}(ds,dy) - \mu_{\theta_2}(dy)ds\right).\end{aligned}$$

Observe that $\{M_t, \mathcal{F}_t\}_{t\geq 0}$ is a zero-mean square integrable martingale since

$$\mathrm{E}_\theta[M_t^2] = -\sigma^2\frac{e^{-2\theta_1 t} - 1}{2\theta_1} - \frac{e^{-2\theta_1 t} - 1}{2\theta_1}\rho_2 \leq \frac{\sigma^2 + \rho_2}{2\theta_1} < \infty,$$

for all $t \geq 0$. Hence, the martingale convergence theorem implies that M_t converges almost surely to the random variable

$$M_\infty = \sigma\int_0^\infty e^{-\theta_1 s}dB_s + \int_0^\infty\int_{\mathbb{R}_0} e^{-\theta_1 s}y\left(N_{\theta_2}(ds,dy) - \mu_{\theta_2}(dy)ds\right),$$

as $t \to \infty$. Thus, $e^{-\theta_1 t}X_t$ converges almost surely to $X_0 + \frac{\rho_1}{\theta_1} + M_\infty$ as $t \to \infty$, which implies that $e^{-2\theta_1 t}X_t^2$ converges almost surely to $\left(X_0 + \frac{\rho_1}{\theta_1} + M_\infty\right)^2$ as $t \to \infty$. Using the integral version of the Toeplitz lemma, one gets that

$$\frac{\int_0^t X_s^2 ds}{\int_0^t e^{2\theta_1 s}ds} \xrightarrow{a.s.} \left(X_0 + \frac{\rho_1}{\theta_1} + M_\infty\right)^2,$$

as $t \to \infty$, which implies that as $t \to \infty$

$$e^{-2\theta_1 t}\int_0^t X_s^2 ds \xrightarrow{a.s.} \frac{1}{2\theta_1}\left(X_0 + \frac{\rho_1}{\theta_1} + M_\infty\right)^2.$$

Hence, taking $\varphi_{1,T}(\theta) = e^{-\theta_1 T}$, that is, $\varphi_T(\theta) = \mathrm{diag}(e^{-\theta_1 T}, \frac{1}{\sqrt{T}}, \ldots, \frac{1}{\sqrt{T}})$, and from (6.4), we get that condition **(A3)** is satisfied with

$$\Gamma_1(\theta) = \begin{pmatrix} \frac{1}{2\sigma^2\theta_1}\left(X_0 + \frac{\rho_1}{\theta_1} + M_\infty\right)^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Assuming additionally condition **(H2)** and applying Theorem 2.3, the LAMN property holds with $\varphi_T(\theta) = \mathrm{diag}(e^{-\theta_1 T}, \frac{1}{\sqrt{T}}, \ldots, \frac{1}{\sqrt{T}})$ and asymptotic Fisher information matrix $\Gamma(\theta) = \Gamma_1(\theta) + \Gamma_2(\theta)$, which is a positive definite matrix.

Case 3: $\sigma > 0$, $\theta_1 < 0$. Recall that if

$$\int_{|y|>2} \log|y| \mu_{\theta_2}(dy) < \infty, \tag{6.5}$$

for all $\theta_2 \in \tilde{\Theta}$, X is ergodic with a unique invariant probability measure $\pi_\theta(dx)$ which can be calculated explicitly (see [20, Theorem 17.5 and Corollary 17.9] and [16, Theorem 2.6]). Therefore, by ergodicity, we have that as $T \to \infty$,

$$\frac{1}{T}\int_0^T X_t^2 dt \xrightarrow{\mathrm{P}_\theta} \int_{\mathbb{R}} x^2 \pi_\theta(dx). \tag{6.6}$$

Proceeding as above, we have that X_t converges almost surely to the random variable

$$X_\infty = -\frac{\rho_1}{\theta_1} + \sigma \int_0^\infty e^{\theta_1 s} dB_s + \int_0^\infty \int_{\mathbb{R}_0} e^{\theta_1 s} y \left(N_{\theta_2}(ds, dy) - \mu_{\theta_2}(dy) ds\right),$$

as $t \to \infty$. Again, by ergodicity, we have that as $T \to \infty$,

$$\frac{1}{T}\int_0^T X_t^2 dt \xrightarrow{\mathrm{P}_\theta} \mathrm{E}_\theta\left[X_\infty^2\right].$$

Notice that

$$\mathrm{E}_\theta\left[X_\infty^2\right] = \lim_{t\to\infty} \mathrm{E}_\theta\left[X_t^2\right] = \frac{2\rho_1^2}{\theta_1^2} - \frac{\sigma^2 + \rho_2}{2\theta_1}.$$

Hence, we deduce that

$$\int_{\mathbb{R}} x^2 \pi_\theta(dx) = \frac{2\rho_1^2}{\theta_1^2} - \frac{\sigma^2 + \rho_2}{2\theta_1}.$$

Then, assuming conditions **(H1)** and **(H2)** and applying Theorem 5.3, the LAN property is satisfied with rate of convergence $\sqrt{T}$ and asymptotic Fisher information matrix

$$\Gamma(\theta) = \begin{pmatrix} \frac{1}{\sigma^2}\left(\frac{2\rho_1^2}{\theta_1^2} - \frac{\sigma^2 + \rho_2}{2\theta_1}\right) & 0 \\ 0 & \tilde{\Gamma}_2(\theta_2) \end{pmatrix}.$$

Case 4: $\sigma > 0$, $\theta_1 = 0$.

Proceeding as above, one obtains the LAN property with rate $\sqrt{T}$ and asymptotic Fisher information $\tilde{\Gamma}_2(\theta_2)$.

Next, we are going to study the asymptotic properties of the maximum likelihood estimator (MLE) of the parameter θ for the parametric model (6.1) in the case that $\sigma > 0$ and $\theta_1 \neq 0$. In particular, we show that the MLE of θ_1 is asymptotically efficient.

Note that the MLE $\hat{\theta}_T = (\hat{\theta}_{1,T}, \hat{\theta}_{2,T})$ of θ satisfies the following equation

$$\nabla_\theta \ell_T(\theta) = \int_0^T \frac{X_t}{\sigma} dW_t + \int_0^T \int_{\mathbb{R}_0} \nabla_\theta \ln(f(\theta_2, y)) (N_{\theta_2}(dt, dy) - \mu_{\theta_2}(dy)dt) = 0,$$

where the process $W = (W_t, 0 \le t \le T)$ is an $(\mathcal{F}_t, 0 \le t \le T)$-Brownian motion under P_θ. Moreover, under P_θ equation (6.1) writes as

$$X_t = X_0 + \theta_1 \int_0^t X_s ds + \sigma W_t + \int_0^t \int_{\mathbb{R}_0} y N_{\theta_2}(ds, dy).$$

First observe that the MLE $\hat{\theta}_{1,T}$ of the drift parameter θ_1 satisfies the following equation

$$\frac{1}{\sigma} \int_0^T X_t dW_t = \frac{1}{\sigma^2} \int_0^T X_t \left(dX_t - \theta_1 X_t dt - \int_{\mathbb{R}_0} y N_{\theta_2}(dt, dy) \right) = 0,$$

which yields that under P_θ,

$$\hat{\theta}_{1,T} = \frac{\int_0^T X_t dX_t - \int_0^T \int_{\mathbb{R}_0} X_t y N_{\theta_2}(dt, dy)}{\int_0^T X_t^2 dt} = \theta_1 + \frac{\sigma \int_0^T X_t dW_t}{\int_0^T X_t^2 dt}.$$

When $\theta_1 > 0$, set $\Gamma_{1,1}(\theta) := \dfrac{1}{2\sigma^2 \theta_1} \left(X_0 + \frac{\rho_1}{\theta_1} + M_\infty \right)^2$. We then have that as $T \to \infty$,

$$e^{\theta_1 T} \left(\hat{\theta}_{1,T} - \theta_1 \right) = \frac{\sigma e^{-\theta_1 T} \int_0^T X_t dW_t}{e^{-2\theta_1 T} \int_0^T X_t^2 dt} \xrightarrow{\mathcal{L}(\mathrm{P}_\theta)} \Gamma_{1,1}(\theta)^{-1/2} \mathcal{N}(0, 1).$$

When $\theta_1 < 0$ we assume (6.5) and set $\Gamma_{1,1}(\theta) := \frac{1}{\sigma^2} \left(\dfrac{2\rho_1^2}{\theta_1^2} - \dfrac{\sigma^2 + \rho_2}{2\theta_1} \right)$. Then, as $T \to \infty$,

$$\sqrt{T} \left(\hat{\theta}_{1,T} - \theta_1 \right) = \frac{\sigma \frac{1}{\sqrt{T}} \int_0^T X_t dW_t}{\frac{1}{T} \int_0^T X_t^2 dt} \xrightarrow{\mathcal{L}(\mathrm{P}_\theta)} \mathcal{N}\left(0, \Gamma_{1,1}(\theta)^{-1}\right).$$

Consequently, we conclude that the MLE $\hat{\theta}_{1,T}$ of θ_1 is asymptotically efficient for all $\theta_1 \neq 0$.

Next, notice that the MLE $\hat{\theta}_{2,T}$ of θ_2 satisfies the following equation

$$\int_0^T \int_{\mathbb{R}_0} \nabla_{\theta_2} \ln(f(\theta_2, y)) (N_{\theta_2}(dt, dy) - \mu_{\theta_2}(dy)dt) = 0, \tag{6.7}$$

which states that the solution $\hat{\theta}_{2,T}$ depends on the Lévy density f. We shall consider here the following two particular cases.

When $(\int_0^t \int_{\mathbb{R}_0} y N_{\theta_2}(ds, dy))_{t \ge 0}$ is a Poisson process $(N_t)_{t \ge 0}$ with intensity $\theta_2 > 0$. In this case, the solution to equation (6.7) is given by

$$\hat{\theta}_{2,T} = \frac{N_T}{T}.$$

By the Central Limit theorem, as $T \to \infty$,

$$\sqrt{T} \left(\hat{\theta}_{2,T} - \theta_2 \right) = \sqrt{T} \left(\frac{N_T}{T} - \theta_2 \right) \xrightarrow{\mathcal{L}(\mathrm{P}_\theta)} \mathcal{N}(0, \theta_2).$$

Thus, in this case, as $T \to \infty$, the MLE $\hat{\theta}_T$ satisfies

$$\varphi_T^{-1}(\theta)\left(\hat{\theta}_T - \theta\right) \overset{\mathcal{L}(\mathrm{P}_\theta)}{\longrightarrow} \Gamma(\theta)^{-1/2}\mathcal{N}\left(0, I_2\right),$$

with

$$\Gamma(\theta) = \begin{pmatrix} \Gamma_{1,1}(\theta) & 0 \\ 0 & \frac{1}{\theta_2} \end{pmatrix},$$

which implies that $\hat{\theta}_T$ is asymptotically efficient. Moreover, $\hat{\theta}_T$ is regular since for all $\theta \in \mathbb{R}_0 \times \tilde{\Theta}$,

$$\varphi_T^{-1}(\theta)\left(\hat{\theta}_T - \theta\right) = \varphi_T^{-1}(\theta)\left\langle \nabla_\theta \ell(\theta)\right\rangle_T^{-1} \nabla_\theta \ell_T(\theta).$$

We next consider the case where the Lévy density $f(\theta_2, y)$ takes the form $\dfrac{\lambda}{\alpha} e^{-y/\alpha}\mathbf{1}_{(0,\infty)}(y)$ with $\lambda, \alpha > 0$ and $\theta_2 = (\lambda, \alpha)$. In this case, solving equation (6.7), we find that the MLE $\hat{\theta}_{2,T} = (\hat{\lambda}_T, \hat{\alpha}_T)$ is given by

$$\hat{\lambda}_T = \frac{N_T}{T}, \quad \text{and} \quad \hat{\alpha}_T = \frac{\int_0^T \int_{\mathbb{R}_0} y N_{\theta_2}(dt, dy)}{N_T},$$

where N_t is a Poisson process with intensity λ. From the Central Limit theorem, we have that as $T \to \infty$,

$$\sqrt{T}\left(\hat{\lambda}_T - \lambda\right) \overset{\mathcal{L}(\mathrm{P}_\theta)}{\longrightarrow} \mathcal{N}\left(0, \lambda\right).$$

Moreover, applying the Central Limit theorem 4.1, we obtain that as $T \to \infty$,

$$\sqrt{T}\left(\hat{\alpha}_T - \alpha\right) = \frac{T}{N_T}\frac{1}{\sqrt{T}}\int_0^T \int_{\mathbb{R}_0} (y - \alpha)(N_{\theta_2}(dt, dy) - \mu_{\theta_2}(dy)dt) \overset{\mathcal{L}(\mathrm{P}_\theta)}{\longrightarrow} \mathcal{N}\left(0, \frac{\alpha^2}{\lambda}\right).$$

Hence, we conclude that as $T \to \infty$,

$$\varphi_T^{-1}(\theta)\left(\hat{\theta}_T - \theta\right) \overset{\mathcal{L}(\mathrm{P}_\theta)}{\longrightarrow} \Gamma(\theta)^{-1/2}\mathcal{N}\left(0, I_3\right),$$

with

$$\Gamma(\theta) = \begin{pmatrix} \Gamma_{1,1}(\theta) & 0 & 0 \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & \frac{\lambda}{\alpha^2} \end{pmatrix},$$

which implies that $\hat{\theta}_T$ is asymptotically efficient. Moreover, $\hat{\theta}_T$ is regular since for all $\theta \in \mathbb{R}_0 \times \tilde{\Theta}$,

$$\varphi_T^{-1}(\theta)\left(\hat{\theta}_T - \theta\right) = \varphi_T^{-1}(\theta)\left\langle \nabla_\theta \ell(\theta)\right\rangle_T^{-1} \nabla_\theta \ell_T(\theta) + o_{\mathrm{P}_\theta}(1).$$

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References

[1] Applebaum, D. (2009), *Lévy Processes and Stochastic Calculus*, Cambridge.

[2] Clément, E., Delattre, S. and Gloter, A. (2013), Asymptotic lower bounds in estimating jumps, *to appear in Bernoulli*.

[3] Feigin, P.D. (1985), Stable convergence of semimartingales, *Stochastic Processes and their Applications*, **19**, 125-134.

[4] Gloter, A. and Gobet, E. (2008), LAMN property for hidden processes: The case of integrated diffusions, *Annales de l'Institut Henri Poincaré - Probabilités et Statistiques*, **44**, 104-128.

[5] Gobet, E. (2001), Local asymptotic mixed normality property for elliptic diffusions: a Malliavin calculus approach, *Bernoulli*, **7**, 899-912.

[6] Gobet, E. (2002), LAN property for ergodic diffusions with discrete observations, *Ann. I. H. Poincaré*, **38**, 711-737.

[7] Ibragimov, I. A. and Has'minskii, R. Z. (1981), *Statistical estimation - Asymptotic theory*, volume 16 of *Applications of Mathematics*, Springer-Verlag, New York. Translated from the Russian by Samuel Kotz.

[8] Jacod, J. and Mémin, J. (1976), Caractéristiques locales et conditions de continuité absolue pour les semi-martingales, *Z. Wahrscheinlichkeitstheorie. verw. Gebiete*, **35**, 1-37.

[9] Jacod, J. and Shiryaev, A. N. (2002), *Limit Theorems For Stochastic process*, Springer.

[10] Jeganathan, P. (1982), On the asymptotic theory of estimation when the limit of the log-likelihood ratios is mixed normal, *Sankhya Ser*, **44**, 173-212.

[11] Kawai, R. and Masuda, H. (2012), Local asymptotic normality for normal inverse Gaussian Lévy processes with high-frequency sampling, *ESAIM: Probability and Statistics*, **17**, 13-32.

[12] Kutoyants, Y.A. (1984), *Parameter Estimation For Stochastic Processes* , Heldermann Verlag Berlin.

[13] Kutoyants, Y.A. (2004), *Statistical Inference for Ergodic Diffusion Processes*, Springer-Verlag.

[14] Le Cam, L. (1960), Locally asymptotically normal families of distributions, *Univ. California, Publ. Statist*, **3**, 37-98.

[15] Luschgy, H. (1992), Local asymptotic mixed normality for semimartingale experiments, *Probab. Theory Relat. Fields*, **92**, 151-176.

[16] Masuda, H. (2007), Ergodicity and exponential β-mixing bounds for multidimensional diffusions with jumps, *Stochastic Processes and their Applications*, **117**, 35-56.

[17] Masuda, H. (2008), On stability of diffusions with compound-Poisson jumps, *Bulletin of Informatics and Cybernetics*, **40**, 60-74.

[18] Meyn, S.P. and Tweedie, R.L. (1993), Stability of Markovian Processes III: Foster-Lyapunov Criteria for Continuous-Time Processes, *Advances in Applied Probability*, **25**, 518-548.

[19] Prakasa Rao, B.L.S. (1999), *Semimartingales and Their Statistical Inference*, Chapman & Hall/CRC.

[20] Sato, K. (1999), *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge.

[21] Sørensen, M. (1991), Likelihood methods for diffusions with jumps, *Statistical Inference in Stochastic Processes*, Probab. Pure Appl., 6, Dekker, New York, 67-105.

[22] Shimizu, Y. and Yoshida, N. (2006), Estimation of Parameters for Diffusion Processes with Jumps from Discrete Observations, *Statistical Inference for Stochastic Processes*, **9**, 227-277.

[23] Shimizu, Y. (2006), *M*-Estimation for Discretely Observed Ergodic Diffusion Processes with Infinitely many Jumps, *Statistical Inference for Stochastic Processes*, **9**, 179-225.

[24] Woerner, J.H.C. (2003), Local asymptotic normality for the scale parameter of stable processes, *Statistics & Probability Letters*, **63**, 61-65.

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